

FORMAL GEOMETRIC QUANTIZATION II

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ABSTRACT. In this paper we pursue the study of formal geometric quantization of non-compact Hamiltonian manifolds. Our main result is the proof that two quantization process coincide. This fact was obtained by Ma and Zhang in the preprint arXiv:0812.3989 by completely different means.

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In the previous article [21], we have studied some functorial properties of the “formal geometric quantization” process $\mathcal{Q}^{-\infty}$, which is defined on *proper Hamiltonian manifolds*, e.g. *non-compact* Hamiltonian manifolds with *proper* moment map.

There is another way, denoted \mathcal{Q}^Φ , of quantizing proper Hamiltonian manifolds by localizing the index of the Dolbeault Dirac operator on the critical points of the square of the moment map [15, 19, 20].

The main purpose of this paper is to provide a geometric proof that the quantization process $\mathcal{Q}^{-\infty}$ and \mathcal{Q}^Φ coincide. This fact was proved by Ma and Zhang in the recent preprint [15] by completely different means.

Keywords: moment map ; symplectic reduction ; geometric quantization ; transversally elliptic symbol.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let us first recall the definition of the geometric quantization of a smooth and compact Hamiltonian manifold. Then we show two way of extending the notion of geometric quantization to the case of a *non-compact* Hamiltonian manifold.

Let K be a compact connected Lie group, with Lie algebra \mathfrak{k} . In the Kostant-Souriau framework, a Hamiltonian K -manifold (M, Ω, Φ) is pre-quantized if there

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is an equivariant Hermitian line bundle L with an invariant Hermitian connection ∇ such that

$$(1.1) \quad \mathcal{L}(X) - \nabla_{X_M} = i\langle \Phi, X \rangle \quad \text{and} \quad \nabla^2 = -i\Omega,$$

for every $X \in \mathfrak{k}$. Here X_M is the vector field on M defined by $X_M(m) = \frac{d}{dt}e^{-tX}m|_0$.

The data (L, ∇) is also called a Kostant-Souriau line bundle, and $\Phi : M \rightarrow \mathfrak{k}^*$ is the moment map. Remark that conditions (1.1) imply via the equivariant Bianchi formula the relation

$$(1.2) \quad \iota(X_M)\Omega = -d\langle \Phi, X \rangle, \quad X \in \mathfrak{k}.$$

Let us recall the notion of geometric quantization when M is **compact**. Choose a K -invariant almost complex structure J on M which is compatible with Ω in the sense that the symmetric bilinear form $\Omega(\cdot, J\cdot)$ is a Riemannian metric. Let $\bar{\partial}_L$ be the Dolbeault operator with coefficients in L , and let $\bar{\partial}_L^*$ be its (formal) adjoint. The *Dolbeault-Dirac operator* on M with coefficients in L is $D_L = \bar{\partial}_L + \bar{\partial}_L^*$, considered as an elliptic operator from $\mathcal{A}^{0,\text{even}}(M, L)$ to $\mathcal{A}^{0,\text{odd}}(M, L)$. Let $R(K)$ be the representation ring of K .

Definition 1.1. *The geometric quantization of a compact Hamiltonian K -manifold (M, Ω, Φ) is the element $\mathcal{Q}_K(M) \in R(K)$ defined as the equivariant index of the Dolbeault-Dirac operator D_L .*

Let us consider the case of a **proper** Hamiltonian K -manifold M : the manifold is (perhaps) **non-compact** but the moment map $\Phi : M \rightarrow \mathfrak{k}^*$ is supposed to be proper. Under this properness assumption, one defines the *formal geometric quantization* of M as an element $\mathcal{Q}_K^{-\infty}(M)$ that belongs to $R^{-\infty}(K)$ [30, 21]. Let us recall the definition.

Let T be a maximal torus of K . Let \mathfrak{k}^* be the dual of the Lie algebra of T containing the weight lattice $\Lambda^* : \alpha \in \Lambda^*$ if $i\alpha : \mathfrak{t} \rightarrow i\mathbb{R}$ is the differential of a character of T . Let $C_K \subset \mathfrak{k}^*$ be a Weyl chamber, and let $\hat{K} := \Lambda^* \cap C_K$ be the set of dominant weights. The ring of characters $R(K)$ has a \mathbb{Z} -basis $V_\mu^K, \mu \in \hat{K} : V_\mu^K$ is the irreducible representation of K with highest weight μ .

A representation E of K is *admissible* if it has finite K -multiplicities : $\dim(\text{hom}_K(V_\mu^K, E)) < \infty$ for every $\mu \in \hat{K}$. Let $R^{-\infty}(K)$ be the Grothendieck group associated to the K -admissible representations. We have an inclusion map $R(K) \hookrightarrow R^{-\infty}(K)$ and $R^{-\infty}(K)$ is canonically identified with $\text{hom}_{\mathbb{Z}}(R(K), \mathbb{Z})$.

For any $\mu \in \hat{K}$ which is a regular value of moment map Φ , the reduced space (or symplectic quotient) $M_\mu := \Phi^{-1}(K \cdot \mu)/K$ is a *compact* orbifold equipped with a symplectic structure Ω_μ . Moreover $L_\mu := (L|_{\Phi^{-1}(\mu)} \otimes \mathbb{C}_{-\mu})/K_\mu$ is a Kostant-Souriau line orbibundle over (M_μ, Ω_μ) . The definition of the index of the Dolbeault-Dirac operator carries over to the orbifold case, hence $\mathcal{Q}(M_\mu) \in \mathbb{Z}$ is defined. In Section 2.3, we explain how this notion of geometric quantization extends further to the case of singular symplectic quotients. So the integer $\mathcal{Q}(M_\mu) \in \mathbb{Z}$ is well defined for every $\mu \in \hat{K}$: in particular $\mathcal{Q}(M_\mu) = 0$ if $\mu \notin \Phi(M)$.

Definition 1.2. *Let (M, Ω, Φ) be a proper Hamiltonian K -manifold which is pre-quantized by a Kostant-Souriau line bundle L . The formal quantization of (M, Ω, Φ)*

is the element of $R^{-\infty}(K)$ defined by

$$\mathcal{Q}_K^{-\infty}(M) = \sum_{\mu \in \hat{K}} \mathcal{Q}(M_\mu) V_\mu^K.$$

When M is compact, the fact that

$$(1.3) \quad \mathcal{Q}_K(M) = \mathcal{Q}_K^{-\infty}(M)$$

is known as the “quantization commutes with reduction Theorem”. This was conjectured by Guillemin-Sternberg in [9] and was first proved by Meinrenken [17] and Meinrenken-Sjamaar [18]. Other proofs of (1.3) were also given by Tian-Zhang [26] and the author [19]. For complete references on the subject the reader should consult [25, 28].

One of the main feature of the formal geometric quantization $\mathcal{Q}^{-\infty}$ is its stability relatively to the restriction to subgroups.

Theorem 1.3 ([21]). *Let M be a pre-quantized Hamiltonian K -manifold which is proper. Let $H \subset K$ be a closed connected Lie subgroup such that M is still proper as a Hamiltonian H -manifold. Then $\mathcal{Q}_K^{-\infty}(M)$ is H -admissible and we have $\mathcal{Q}_K^{-\infty}(M)|_H = \mathcal{Q}_H^{-\infty}(M)$ in $R^{-\infty}(H)$.*

When M is a proper Hamiltonian K -manifold, we can also define another “formal geometric quantization”, denoted

$$(1.4) \quad \mathcal{Q}_K^\Phi(M) \in R^{-\infty}(K),$$

by localizing the index of the Dolbeault-Dirac operator D_L on the set $\text{Cr}(\|\Phi\|^2)$ of critical points of the square of the moment map (see Section 2.2 for the precise definition). We proved in previous papers [20, 21, 23] that

$$(1.5) \quad \mathcal{Q}_K^{-\infty}(M) = \mathcal{Q}_K^\Phi(M).$$

in some situations:

- M is a coadjoint orbit of a semi-simple Lie group S that parametrizes a representation of the discrete series of S ,
- M is a Hermitian vector space.

In her ICM 2006 plenary lecture [29], Vergne conjectured that (1.5) holds when $\text{Cr}(\|\Phi\|^2)$ is compact. Recently, Ma and Zhang [15] prove the following generalisation of this conjecture.

Theorem 1.4. *The equality (1.5) holds for **any** proper Hamiltonian K -manifold.*

This article is dedicated to the study of the quantization map \mathcal{Q}^Φ :

- In Section 2.2, we give the precise definition of the quantization process \mathcal{Q}^Φ . In particular, we refine the constant c_γ appearing in [15][Theorem 0.1].
- In Section 2.4, we explain how to compute the quantization of a point.
- We give in Section 3 another proof of Theorem 1.4 by using the technique of symplectic cutting developed in [21].
- In Section 4, we consider the case where $K = K_1 \times K_2$ acts on M in a way that the symplectic reduction $M//_0 K_1$ is a *smooth* proper K_2 -Hamiltonian manifold. We show then that the K_1 -invariant part of $\mathcal{Q}_{K_1 \times K_2}^\Phi(M)$ is equal to $\mathcal{Q}_{K_2}^{\Phi_2}(M//_0 K_1)$.

In Section 5, we study the example where M is the cotangent bundle of a homogeneous space: $M = T^*(K/H)$ where H is a closed subgroup of K . We see that $T^*(K/H)$ is a proper Hamiltonian K -manifold prequantized by the trivial line bundle. A direct computation gives

$$(1.6) \quad \mathcal{Q}_K^\Phi(T^*(K/H)) = L^2(K/H) \quad \text{in} \quad R^{-\infty}(K).$$

Let us denote $[T^*(K/H)]_{\mu,K}$ the symplectic reduction at $\mu \in \widehat{K}$ of the K -Hamiltonian manifold $T^*(K/H)$. Theorem 1.4 together with (1.6) give

$$\mathcal{Q}([T^*(K/H)]_{\mu,K}) = \dim[V_\mu^K]^H,$$

for any $\mu \in \widehat{K}$. Here $[V_\mu^K]^H \subset V_\mu^K$ is the subspace of H -invariant vectors.

Then we consider the action of a closed connected subgroup $G \subset K$ on $T^*(K/H)$. We first check that $T^*(K/H)$ is a *proper* Hamiltonian G -manifold if and only if the restriction $L^2(K/H)|_G$ is an admissible G -representation. Then, using Theorem 1.3, we get that

$$(1.7) \quad \mathcal{Q}_G^{-\infty}(T^*(K/H)) = L^2(K/H)|_G \quad \text{in} \quad R^{-\infty}(G).$$

In other words, the multiplicity of V_λ^G in $L^2(K/H)$ is equal to the quantization of the reduced space $[T^*(K/H)]_{\lambda,G}$.

2. QUANTIZATIONS OF NON-COMPACT MANIFOLDS

In this section we define the quantization process \mathcal{Q}^Φ , and we give another definition of the quantization process $\mathcal{Q}^{-\infty}$ that uses the notion of symplectic cutting [21].

2.1. Transversally elliptic symbols. Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah-Singer in [1]. For an axiomatic treatment of the index morphism see Berline-Vergne [6, 7] and Paradan-Vergne [22]. For a short introduction see [19].

Let \mathcal{X} be a *compact* K -manifold. Let $p : T\mathcal{X} \rightarrow \mathcal{X}$ be the projection, and let $(-, -)_\mathcal{X}$ be a K -invariant Riemannian metric. If E^0, E^1 are K -equivariant complex vector bundles over \mathcal{X} , a K -equivariant morphism $\sigma \in \Gamma(T\mathcal{X}, \text{hom}(p^*E^0, p^*E^1))$ is called a *symbol* on \mathcal{X} . The subset of all $(x, v) \in T\mathcal{X}$ where $\sigma(x, v) : E_x^0 \rightarrow E_x^1$ is not invertible is called the *characteristic set* of σ , and is denoted by $\text{Char}(\sigma)$.

In the following, the product of a symbol σ by a complex vector bundle $F \rightarrow M$, is the symbol

$$\sigma \otimes F$$

defined by $\sigma \otimes F(x, v) = \sigma(x, v) \otimes \text{Id}_{F_x}$ from $E_x^0 \otimes F_x$ to $E_x^1 \otimes F_x$. Note that $\text{Char}(\sigma \otimes F) = \text{Char}(\sigma)$.

Let $T_K\mathcal{X}$ be the following subset of $T\mathcal{X}$:

$$T_K\mathcal{X} = \{(x, v) \in T\mathcal{X}, (v, X_\mathcal{X}(x))_x = 0 \text{ for all } X \in \mathfrak{k}\}.$$

A symbol σ is *elliptic* if σ is invertible outside a compact subset of $T\mathcal{X}$ (i.e. $\text{Char}(\sigma)$ is compact), and is *K -transversally elliptic* if the restriction of σ to $T_K\mathcal{X}$ is invertible outside a compact subset of $T_K\mathcal{X}$ (i.e. $\text{Char}(\sigma) \cap T_{K_2}\mathcal{X}$ is compact). An elliptic symbol σ defines an element in the equivariant \mathbf{K} -theory of $T\mathcal{X}$ with compact support, which is denoted by $\mathbf{K}_K(T\mathcal{X})$, and the index of σ is a virtual finite dimensional representation of K , that we denote $\text{Index}_\mathcal{X}^K(\sigma) \in R(K)$ [2, 3, 4, 5].

¹The map $\sigma(x, v)$ will be also denote $\sigma|_x(v)$

Let

$$R_{tc}^{-\infty}(K) \subset R^{-\infty}(K)$$

be the $R(K)$ -submodule formed by all the infinite sum $\sum_{\mu \in \widehat{K}} m_\mu V_\mu^K$ where the map $\mu \in \widehat{K} \mapsto m_\mu \in \mathbb{Z}$ has at most a *polynomial* growth. The $R(K)$ -module $R_{tc}^{-\infty}(K)$ is the Grothendieck group associated to the *trace class* virtual K -representations: we can associate to any $V \in R_{tc}^{-\infty}(K)$, its trace $k \rightarrow \text{Tr}(k, V)$ which is a generalized function on K invariant by conjugation. Then the trace defines a morphism of $R(K)$ -module

$$(2.8) \quad R_{tc}^{-\infty}(K) \hookrightarrow \mathcal{C}^{-\infty}(K)^K.$$

A K -transversally elliptic symbol σ defines an element of $\mathbf{K}_K(\text{T}_K \mathcal{X})$, and the index of σ is defined as a trace class virtual representation of K , that we still denote $\text{Index}_{\mathcal{X}}^K(\sigma) \in R_{tc}^{-\infty}(K)$.

Remark that any elliptic symbol of $\text{T}\mathcal{X}$ is K -transversally elliptic, hence we have a restriction map $\mathbf{K}_K(\text{T}\mathcal{X}) \rightarrow \mathbf{K}_K(\text{T}_K \mathcal{X})$, and a commutative diagram

$$(2.9) \quad \begin{array}{ccc} \mathbf{K}_K(\text{T}\mathcal{X}) & \longrightarrow & \mathbf{K}_K(\text{T}_K \mathcal{X}) \\ \text{Index}_{\mathcal{X}}^K \downarrow & & \downarrow \text{Index}_{\mathcal{X}}^K \\ R(K) & \longrightarrow & R_{tc}^{-\infty}(K) . \end{array}$$

Using the *excision property*, one can easily show that the index map $\text{Index}_{\mathcal{U}}^K : \mathbf{K}_K(\text{T}_K \mathcal{U}) \rightarrow R_{tc}^{-\infty}(K)$ is still defined when \mathcal{U} is a K -invariant relatively compact open subset of a K -manifold (see [19][section 3.1]).

Suppose now that the group K is equal to the product $K_1 \times K_2$. When a symbol σ is $K_1 \times K_2$ -transversally elliptic we will be interested in the K_1 -invariant part of its index, that we denote

$$\left[\text{Index}_{\mathcal{X}}^{K_1 \times K_2}(\sigma) \right]^{K_1} \in R_{tc}^{-\infty}(K_2).$$

An intermediate notion between the “ellipticity” and “ $K_1 \times K_2$ -transversal ellipticity” is the “ K_1 -transversal ellipticity”. When a $K_1 \times K_2$ -equivariant morphism σ is K_1 -transversally elliptic, its index $\text{Index}_{\mathcal{X}}^{K_1 \times K_2}(\sigma) \in R_{tc}^{-\infty}(K_1 \times K_2)$, viewed as a generalized function on $K_1 \times K_2$, is *smooth* relatively to the variable in K_2 . It implies that $\text{Index}_{\mathcal{X}}^{K_1 \times K_2}(\sigma) = \sum_{\lambda} \theta(\lambda) \otimes V_{\lambda}^{K_1}$ with

$$\theta(\lambda) \in R(K_2), \quad \forall \lambda \in \widehat{K_1}.$$

In particular, we know that

$$\left[\text{Index}_{\mathcal{X}}^{K_1 \times K_2}(\sigma) \right]^{K_1} = \theta(0)$$

belongs to $R(K_2)$.

Let us recall the multiplicative property of the index map for the product of manifolds that was proved by Atiyah-Singer in [1]. Consider a compact Lie group K_2 acting on two manifolds \mathcal{X}_1 and \mathcal{X}_2 , and assume that another compact Lie group K_1 acts on \mathcal{X}_1 commuting with the action of K_2 .

The external product of complexes on $\text{T}\mathcal{X}_1$ and $\text{T}\mathcal{X}_2$ induces a multiplication (see [1, 22]):

$$\odot : K_{K_1 \times K_2}(\text{T}_{K_1} \mathcal{X}_1) \times K_{K_2}(\text{T}_{K_2} \mathcal{X}_2) \longrightarrow K_{K_1 \times K_2}(\text{T}_{K_1 \times K_2}(\mathcal{X}_1 \times \mathcal{X}_2)).$$

The following property will be used frequently in the paper.

Theorem 2.1 (Multiplicative property). *For any $[\sigma_1] \in K_{K_1 \times K_2}(\mathbb{T}_{K_1} \mathcal{X}_1)$ and any $[\sigma_2] \in K_{K_2}(\mathbb{T}_{K_2} \mathcal{X}_2)$ we have*

$$\text{Index}_{\mathcal{X}_1 \times \mathcal{X}_2}^{K_1 \times K_2}([\sigma_1] \odot [\sigma_2]) = \text{Index}_{\mathcal{X}_1}^{K_1 \times K_2}([\sigma_1]) \otimes \text{Index}_{\mathcal{X}_2}^{K_2}([\sigma_2]).$$

We will use in this article the notion of *support* of a generalized character.

Definition 2.2. *The support of $\chi := \sum_{\mu \in \hat{K}} a_\mu V_\mu^K \in R^{-\infty}(K)$ is the set of $\mu \in \hat{K}$ such that $a_\mu \neq 0$.*

We will say that $\chi \in R^{-\infty}(K)$ is supported outside $B \subset \mathfrak{t}^*$ if the support of χ does not intersect B . Note that an infinite sum $\sum_{i \in I} \chi_i$ converges in $R^{-\infty}(K)$ if for each ball

$$B_r = \{\xi \in \mathfrak{t}^* \mid \|\xi\| < r\}$$

the set $\{i \in I \mid \text{support}(\chi_i) \cap B_r \neq \emptyset\}$ is finite.

Definition 2.3. *We denote by $O(r)$ any character of $R^{-\infty}(K)$ which is supported outside the ball B_r .*

2.2. Definition and first properties of \mathcal{Q}^Φ . Let (M, Ω, Φ) be a proper Hamiltonian K -manifold prequantized by an equivariant line bundle L . Let J be an invariant almost complex structure compatible with Ω . Let $p : TM \rightarrow M$ be the projection.

Let us first describe the principal symbol of the Dolbeault-Dirac operator $\bar{\partial}_L + \bar{\partial}_L^*$. The complex vector bundle $(T^*M)^{0,1}$ is K -equivariantly identified with the tangent bundle TM equipped with the complex structure J . Let h be the Hermitian structure on (TM, J) defined by : $h(v, w) = \Omega(v, Jw) - i\Omega(v, w)$ for $v, w \in TM$. The symbol

$$\text{Thom}(M, J) \in \Gamma(M, \text{hom}(p^*(\wedge_{\mathbb{C}}^{\text{even}} TM), p^*(\wedge_{\mathbb{C}}^{\text{odd}} TM)))$$

at $(m, v) \in TM$ is equal to the Clifford map

$$(2.10) \quad \mathbf{c}_m(v) : \wedge_{\mathbb{C}}^{\text{even}} T_m M \longrightarrow \wedge_{\mathbb{C}}^{\text{odd}} T_m M,$$

where $\mathbf{c}_m(v).w = v \wedge w - \iota(v)w$ for $w \in \wedge_{\mathbb{C}}^{\bullet} T_m M$. Here $\iota(v) : \wedge_{\mathbb{C}}^{\bullet} T_m M \rightarrow \wedge_{\mathbb{C}}^{\bullet-1} T_m M$ denotes the contraction map relative to h . Since $\mathbf{c}_m(v)^2 = -\|v\|^2 \text{Id}$, the map $\mathbf{c}_m(v)$ is invertible for all $v \neq 0$. Hence the characteristic set of $\text{Thom}(M, J)$ corresponds to the 0-section of TM .

It is a classical fact that the principal symbol of the Dolbeault-Dirac operator $\bar{\partial}_L + \bar{\partial}_L^*$ is equal to²

$$(2.11) \quad \text{Thom}(M, J) \otimes L,$$

see [8]. Here also we have $\text{Char}(\text{Thom}(M, J) \otimes L) = 0$ – section of TM .

Remark 2.4. *When the manifold M is a product $M_1 \times M_2$ the symbol $\text{Thom}(M, J) \otimes L$ is equal to the product $\sigma_1 \odot \sigma_2$ where $\sigma_k = \text{Thom}(M_k, J_k) \otimes L_k$.*

²Here we use an identification $T^*M \simeq TM$ given by an invariant Riemannian metric.

When M is compact, the symbol $\text{Thom}(M, J) \otimes L$ is elliptic and then defines an element of the equivariant \mathbf{K} -group of TM . The topological index of $\text{Thom}(M, J) \otimes L \in \mathbf{K}_K(TM)$ is equal to the analytical index of the Dolbeault-Dirac operator $\bar{\partial}_L + \bar{\partial}_L^*$:

$$(2.12) \quad \mathcal{Q}_K(M) = \text{Index}_M^K(\text{Thom}(M, J) \otimes L) \quad \text{in} \quad R(K).$$

When M is not compact the topological index of $\text{Thom}(M, J) \otimes L$ is not defined. In order to extend the notion of geometric quantization to this setting we deform the symbol $\text{Thom}(M, J) \otimes L$ in the ‘‘Witten’’ way [19, 20]. Consider the identification $\xi \mapsto \tilde{\xi}, \mathfrak{k}^* \rightarrow \mathfrak{k}$ defined by a K -invariant scalar product on \mathfrak{k}^* . We define the *Kirwan vector field* on M :

$$(2.13) \quad \kappa_m = \left(\widetilde{\Phi(m)} \right)_M(m), \quad m \in M.$$

Definition 2.5. *The symbol $\text{Thom}(M, J) \otimes L$ pushed by the vector field κ is the symbol \mathbf{c}^κ defined by the relation*

$$\mathbf{c}^\kappa|_m(v) = \text{Thom}(M, J) \otimes L|_m(v - \kappa_m)$$

for any $(m, v) \in TM$.

Note that $\mathbf{c}^\kappa|_m(v)$ is invertible except if $v = \kappa_m$. If furthermore v belongs to the subset $T_K M$ of tangent vectors orthogonal to the K -orbits, then $v = 0$ and $\kappa_m = 0$. Indeed κ_m is tangent to $K \cdot m$ while v is orthogonal.

Since κ is the Hamiltonian vector field of the function $\frac{-1}{2}\|\Phi\|^2$, the set of zeros of κ coincides with the set $\text{Cr}(\|\Phi\|^2)$ of critical points of $\|\Phi\|^2$. Finally we have

$$\text{Char}(\mathbf{c}^\kappa) \cap T_K M \simeq \text{Cr}(\|\Phi\|^2).$$

In general $\text{Cr}(\|\Phi\|^2)$ is not compact, so \mathbf{c}^κ does not define a transversally elliptic symbol on M . In order to define a kind of index of \mathbf{c}^κ , we proceed as follows. For any invariant open relatively compact subset $U \subset M$ the set $\text{Char}(\mathbf{c}^\kappa|_U) \cap T_K U \simeq \text{Cr}(\|\Phi\|^2) \cap U$ is compact when

$$(2.14) \quad \partial U \cap \text{Cr}(\|\Phi\|^2) = \emptyset.$$

When (2.14) holds we denote

$$(2.15) \quad \mathcal{Q}_K^\Phi(U) := \text{Index}_U^K(\mathbf{c}^\kappa|_U) \quad \in \quad R_{tc}^{-\infty}(K)$$

the equivariant index of the transversally elliptic symbol $\mathbf{c}^\kappa|_U$.

It will be useful to understand the dependance of the generalized character $\mathcal{Q}_K^\Phi(U)$ relatively to the data (U, Ω, L) . So we consider two proper Hamiltonian K -manifolds (M, Ω, Φ) and (M', Ω', Φ') respectively prequantized by the line bundles L and L' . Let $V \subset M$ and $V' \subset M'$ two invariant open subsets.

Proposition 2.6. • *The generalized character $\mathcal{Q}_K^\Phi(U)$ does not depend of the choice of an invariant almost complex structure on U which is compatible with $\Omega|_U$.*

• *Suppose that there exists an equivariant diffeomorphism $\Psi : V \rightarrow V'$ such that*

- (1) $\Psi^*(\Phi') = \Phi$,
- (2) $\Psi^*(L') = L$,
- (3) *there exists an homotopy of symplectic forms taking $\Psi^*(\Omega'|_{V'})$ to $\Omega|_V$.*

Let $U' \subset \overline{U'} \subset V'$ be an invariant open relatively compact subset such that $\partial U'$ satisfies (2.14). Take $U = \Psi^{-1}(U')$. Then ∂U satisfies (2.14) and

$$\mathcal{Q}_K^{\Phi'}(U') = \mathcal{Q}_K^{\Phi}(U) \in R^{-\infty}(K).$$

Proof. Let us prove the first point. Let $\mathbf{c}_i^{\kappa}|_U, i = 0, 1$ be the transversally elliptic symbols defined with the compatible almost complex structure $J_i, i = 0, 1$. Since the space of compatible almost complex structure is contractible, there exist an homotopy $J_t, t \in [0, 1]$ of almost complex structures linking J_0 and J_1 . If we use Lemma 2.2 in [19], we know that there exists an invertible bundle map $A \in \Gamma(U, \text{End}(TU))$, homotopic to the identity, such that $A \circ J_0 = J_1 \circ A$. With the help of A we prove then that the symbols $\mathbf{c}_0^{\kappa}|_U$ and $\mathbf{c}_1^{\kappa}|_U$ define the same class in $\mathbf{K}_K(T_K U)$ (see [19][Lemma 2.2]). Hence their equivariant index coincide.

Let us prove the second point. The characters $\mathcal{Q}_K^{\Phi}(U)$ and $\mathcal{Q}_K^{\Phi'}(U')$ are computed as the equivariant index of the symbols $\mathbf{c}^{\kappa}|_U$ and $\mathbf{c}^{\kappa'}|_{U'}$. Let $\tilde{\mathbf{c}}^{\kappa}|_U$ the pull back of $\mathbf{c}^{\kappa'}|_{U'}$ by Ψ . Thanks to the point (1) and (2), the only thing which differs in the definitions of the symbols $\mathbf{c}^{\kappa}|_U$ and $\tilde{\mathbf{c}}^{\kappa}|_U$ are the almost complex structures J and $\tilde{J} = \Psi^*(J')$: the first one is compatible with Ω and the second one with $\Psi^*(\Omega'|_{V'})$. Since these two symplectic structure are homotopic, one sees that the almost complex structures J and \tilde{J} are also homotopic. So we can conclude like in the first point. \square

Let us recall the basic fact concerning the singular values of $\|\Phi\|^2$.

Lemma 2.7. *The set of singular values of $\|\Phi\|^2 : M \rightarrow \mathbb{R}$ forms a sequence $0 \leq r_1 < r_2 < \dots < r_k < \dots$ which is finite iff $\text{Cr}(\|\Phi\|^2)$ is compact. In the other case $\lim_{k \rightarrow \infty} r_k = \infty$.*

At each regular value R of $\text{Cr}(\|\Phi\|^2)$, we associate the invariant open subset $M_{<R} := \{\|\Phi\|^2 < R\}$ which satisfies (2.14). The restriction $\mathbf{c}^{\kappa}|_{M_{<R}}$ defines then a transversally elliptic symbol on $M_{<R}$: let $\mathcal{Q}_K^{\Phi}(M_{<R})$ be its equivariant index.

Let us show that $\mathcal{Q}_K^{\Phi}(M_{<R})$ has a limit when $R \rightarrow \infty$. The set $\text{Cr}(\|\Phi\|^2)$ has the following decomposition

$$(2.16) \quad \text{Cr}(\|\Phi\|^2) = \bigcup_{\beta \in \mathcal{B}} \underbrace{K \cdot (M^{\tilde{\beta}} \cap \Phi^{-1}(\beta))}_{Z_{\beta}}$$

where the \mathcal{B} is a subset of the Weyl chamber \mathfrak{t}_+^* . Note that each part Z_{β} is compact, hence \mathcal{B} is finite only if $\text{Cr}(\|\Phi\|^2)$ is compact. When $\text{Cr}(\|\Phi\|^2)$ is non-compact, the set \mathcal{B} is infinite, but it is easy to see that $\mathcal{B} \cap B_r$ is finite for any $r \geq 0$. For any $\beta \in \mathcal{B}$, we consider a relatively compact open invariant neighborhood \mathcal{U}_{β} of Z_{β} such that $\text{Cr}(\|\Phi\|^2) \cap \overline{\mathcal{U}_{\beta}} = Z_{\beta}$.

Definition 2.8. *We denote*

$$\mathcal{Q}_K^{\beta}(M) \in R_{tc}^{-\infty}(K)$$

the index³ of the transversally elliptic symbol $\mathbf{c}^{\kappa}|_{\mathcal{U}_{\beta}}$.

A simple application of the excision property [19] gives that

$$(2.17) \quad \mathcal{Q}_K^{\Phi}(M_{<R}) := \sum_{\|\beta\|^2 < R} \mathcal{Q}_K^{\beta}(M).$$

³The index of $\mathbf{c}^{\kappa}|_{\mathcal{U}_{\beta}}$ was denoted $RR_{\beta}^K(M, L)$ in [19].

We have now the key fact

Theorem 2.9. *The generalized character $\mathcal{Q}_K^\beta(M)$ is supported outside the open ball $B_{\|\beta\|}$.*

Proof. Proposition 2.9 follows directly from the computations done in [19]. First consider the case where $\beta \neq 0$ is a K -invariant element of \mathcal{B} . Let $i : \mathbb{T}_\beta \hookrightarrow T$ be the compact torus generated by β . If F is \mathbb{Z} -module we denote $F \hat{\otimes} R^{-\infty}(\mathbb{T}_\beta)$ the \mathbb{Z} -module formed by the infinite formal sums $\sum_a E_a h^a$ taken over the set of weights of \mathbb{T}_β , where $E_a \in F$ for every a .

Since \mathbb{T}_β lies in the center of K , the morphism $\pi : (k, t) \in K \times \mathbb{T}_\beta \mapsto kt \in K$ induces a map $\pi^* : R^{-\infty}(K) \rightarrow R^{-\infty}(K) \hat{\otimes} R^{-\infty}(\mathbb{T}_\beta)$.

The normal bundle \mathcal{N} of $M^{\tilde{\beta}}$ in M inherits a canonical complex structure $J_{\mathcal{N}}$ on the fibers. We denote by $\overline{\mathcal{N}} \rightarrow M^{\tilde{\beta}}$ the complex vector bundle with the opposite complex structure. The torus \mathbb{T}_β is included in the center of K , so the bundle $\overline{\mathcal{N}}$ and the virtual bundle $\wedge_{\mathbb{C}}^{\bullet} \overline{\mathcal{N}} := \wedge_{\mathbb{C}}^{\text{even}} \overline{\mathcal{N}} \xrightarrow{0} \wedge_{\mathbb{C}}^{\text{odd}} \overline{\mathcal{N}}$ carry a $K \times \mathbb{T}_\beta$ -action: they can be considered as elements of $K_{K \times \mathbb{T}_\beta}(M^{\tilde{\beta}}) = K_K(M^{\tilde{\beta}}) \otimes R(\mathbb{T}_\beta)$.

In [19], we have defined an inverse of $\wedge_{\mathbb{C}}^{\bullet} \overline{\mathcal{N}}$, $[\wedge_{\mathbb{C}}^{\bullet} \overline{\mathcal{N}}]_{\beta}^{-1} \in K_K(M^{\tilde{\beta}}) \hat{\otimes} R^{-\infty}(\mathbb{T}_\beta)$, which is polarized by β . It means that $[\wedge_{\mathbb{C}}^{\bullet} \overline{\mathcal{N}}]_{\beta}^{-1} = \sum_a N_a h^a$ with $N_a \neq 0$ only if $\langle a, \beta \rangle \geq 0$.

We prove in [19] the following localization formula :

$$(2.18) \quad \pi^* \left[\mathcal{Q}_K^\beta(M) \right] = R R_{\beta}^{K \times \mathbb{T}_\beta} \left(M^{\tilde{\beta}}, L|_{M^{\tilde{\beta}}} \otimes [\wedge_{\mathbb{C}}^{\bullet} \overline{\mathcal{N}}]_{\beta}^{-1} \right),$$

as an equality in $R^{-\infty}(K) \hat{\otimes} R^{-\infty}(\mathbb{T}_\beta)$. With (2.18) in hand, it is easy to see that V_{μ}^K occurs in the character $\mathcal{Q}_K^\beta(M)$ only if $(\mu, \beta) \geq \|\beta\|^2$ (See Lemma 9.4 in [19]).

Now we consider the case where $\beta \in \mathcal{B}$ is not a K -invariant element. Let σ be the unique open face of the Weyl chamber \mathfrak{t}_+^* which contains β . Let K_σ be the corresponding stabilizer subgroup. We consider the symplectic slice $\mathcal{Y}_\sigma \subset M$: it is a K_σ invariant Hamiltonian submanifold of M which is prequantized by the line bundle $L|_{\mathcal{Y}_\sigma}$. The restriction of Φ to \mathcal{Y}_σ is a moment map $\Phi_\sigma : \mathcal{Y}_\sigma \rightarrow \mathfrak{k}_\sigma^*$ which is proper in a neighborhood of $\beta \in \mathfrak{k}_\sigma^*$. The set

$$K_\sigma \cdot (\mathcal{Y}_\sigma^{\tilde{\beta}} \cap \Phi_\sigma^{-1}(\beta)) = M^{\tilde{\beta}} \cap \Phi^{-1}(\beta)$$

is a component of $\text{Cr}(\|\Phi_\sigma\|^2)$. Let $\mathcal{Q}_{K_\sigma}^\beta(\mathcal{Y}_\sigma) \in R_{tc}^{-\infty}(K_\sigma)$ be the corresponding character (see Definition 2.8).

We prove in [19][Section 7], the following induction formula:

$$(2.19) \quad \mathcal{Q}_K^\beta(M) = \text{Hol}_{K_\sigma}^K \left(\mathcal{Q}_{K_\sigma}^\beta(\mathcal{Y}_\sigma) \right)$$

where $\text{Hol}_{K_\sigma}^K : R^{-\infty}(K_\sigma) \rightarrow R^{-\infty}(K)$ is the holomorphic induction map. See the Appendix in [19] for the definition and properties of these induction maps.

We know from the previous case that

$$\mathcal{Q}_{K_\sigma}^\beta(\mathcal{Y}_\sigma) = \sum_{\mu \in \widehat{K_\sigma}} m_\mu V_\mu^{K_\sigma}$$

where $m_\mu \neq 0 \implies (\mu, \beta) \geq \|\beta\|^2$. Then, with (2.19), we get

$$\begin{aligned} \mathcal{Q}_{K_\sigma}^\beta(\mathcal{Y}_\sigma) &= \sum_{(\mu, \beta) \geq \|\beta\|^2} m_\mu \operatorname{Hol}_{K_\sigma}^K(V_\mu^{K_\sigma}) \\ &= \sum_{(\mu, \beta) \geq \|\beta\|^2} m_\mu \operatorname{Hol}_T^K(t^\mu), \end{aligned}$$

where $\operatorname{Hol}_T^K : R^{-\infty}(T) \rightarrow R^{-\infty}(K)$ is the holomorphic induction map.

Let ρ be half the sum of the positive roots. The term $\operatorname{Hol}_T^K(t^\mu)$ is equal to 0 when $\mu + \rho$ is not a regular element of \mathfrak{t}^* . When $\mu + \rho$ is a regular element of \mathfrak{t}^* , we have $\operatorname{Hol}_T^K(t^\mu) = (-1)^{|\omega|} V_{\mu_\omega}^K$ where

$$\mu_\omega = \omega(\mu + \rho) - \rho$$

is dominant for a unique $\omega \in W$.

Finally, a representation V_λ^K appears in the character $\mathcal{Q}_K^\beta(M)$ only if $\lambda = \mu_\omega$ for a weight μ satisfying $(\mu, \beta) \geq \|\beta\|^2$. Hence, for such λ , we have

$$\begin{aligned} \|\lambda\| &= \|\mu + \rho - \omega^{-1}\rho\| \\ &\geq (\mu + \rho - \omega^{-1}\rho, \frac{\beta}{\|\beta\|}) \\ &\geq \|\beta\|. \end{aligned}$$

In the last inequality we use that $(\rho - \omega^{-1}\rho, \beta) \geq 0$ since $\rho - \omega^{-1}\rho$ is a sum of positive roots, and $\beta \in \mathfrak{t}_+^*$. □

With the help of Theorem 2.9 and decomposition (2.17), we see that the multiplicity of V_γ^K in $\mathcal{Q}_K^\Phi(M_{<R})$ does not depend on the regular value $R > \|\gamma\|^2$. We can refine the constant c_γ appearing in [15][Theorem 0.1]: take c_γ equal to $\|\gamma\|^2$ instead of⁴ $\|\gamma + \rho\|^2 - \|\rho\|^2 \geq \|\gamma\|^2$.

Definition 2.10. *The generalized character $\mathcal{Q}_K^\Phi(M)$ is defined as the limit in $R^{-\infty}(K)$ of $\mathcal{Q}_K^\Phi(M_{<R})$ when R goes to infinity. In other words*

$$(2.20) \quad \mathcal{Q}_K^\Phi(M) = \sum_{\beta \in \mathcal{B}} \mathcal{Q}_K^\beta(M).$$

Note that for any regular value R of $\|\Phi\|^2$ we have the useful relation

$$(2.21) \quad \mathcal{Q}_K^\Phi(M) = \mathcal{Q}_K^\Phi(M_{<R}) + O(\sqrt{R}).$$

2.3. Quantization of a symplectic quotient. We will now explain how we define the geometric quantization of *singular* compact Hamiltonian manifolds : here “singular” means that the manifold is obtained by symplectic reduction.

Let (N, Ω) be a smooth symplectic manifold equipped with a Hamiltonian action of $K_1 \times K_2$: we denote $(\Phi_1, \Phi_2) : N \rightarrow \mathfrak{k}_1^* \times \mathfrak{k}_2^*$ the corresponding moment map. We assume that N is pre-quantized by a $K_1 \times K_2$ -equivariant line bundle L and we

⁴Here ρ is half the sum of the positive roots. Hence $\|\gamma + \rho\|^2 - \|\rho\|^2 - \|\gamma\|^2 = 2(\rho, \gamma) \geq 0$ and $(\rho, \gamma) = 0$ only if the weight γ belongs to the center of $\mathfrak{k} \simeq \mathfrak{k}^*$.

suppose that the map Φ_1 is **proper**. One wants to define the geometric quantization of the (compact) symplectic quotient

$$N//_0 K_1 := \Phi_1^{-1}(0)/K_1.$$

Let κ_1 be the Kirwan vector field attached to the moment map Φ_1 . We denote by \mathbf{c}^{κ_1} the symbol $\text{Thom}(N, J) \otimes L$ pushed by the vector field κ_1 . For any regular value R_1 of $\|\Phi_1\|^2$, we consider the restriction $\mathbf{c}^{\kappa_1}|_{N_{<R_1}}$ to the invariant, open subset $N_{<R_1} := \{\|\Phi_1\|^2 < R_1\}$. The symbol $\mathbf{c}^{\kappa_1}|_{N_{<R_1}}$ is $K_1 \times K_2$ -equivariant and K_1 -transversally elliptic, hence we can consider its index

$$\text{Index}_{N_{<R_1}}^{K_1 \times K_2}(\mathbf{c}^{\kappa_1}|_{N_{<R_1}}) \in R^{-\infty}(K_1 \times K_2).$$

which is smooth relatively to the parameter in K_2 . We consider the following extension of Definition 2.10.

Definition 2.11. *The generalized character $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(N)$ is defined as the limit in $R^{-\infty}(K_1 \times K_2)$ of $\text{Index}_{N_{<R_1}}^{K_1 \times K_2}(\mathbf{c}^{\kappa_1}|_{N_{<R_1}})$ when R_1 goes to infinity.*

Here $\text{Cr}(\|\Phi_1\|^2)$ is equal to the disjoint union of the compact $K_1 \times K_2$ -invariant subsets $Z_{\beta_1} := K_1 \cdot (M^{\beta_1} \cap \Phi_1^{-1}(\beta_1))$, $\beta_1 \in \mathcal{B}_1$. For $\beta_1 \in \mathcal{B}_1$, we consider an invariant relatively compact open subset \mathcal{U}_{β_1} such that: $Z_{\beta_1} \subset \mathcal{U}_{\beta_1}$ and $Z_{\beta_1} = \text{Cr}(\|\Phi_1\|^2) \cap \overline{\mathcal{U}_{\beta_1}}$. Let $\mathcal{Q}_{K_1 \times K_2}^{\beta_1}(N) \in R^{-\infty}(K_1 \times K_2)$ be the equivariant index of the K_1 -transversally elliptic symbol $\mathbf{c}_{\kappa_1}^{L_1}|_{\mathcal{U}_{\beta_1}}$. The K_1 -transversality condition imposes that $\mathcal{Q}_{K_1 \times K_2}^{\beta_1}(N) = \sum_{\lambda} \theta^{\beta_1}(\lambda) \otimes V_{\lambda}^{K_1}$ with

$$\theta^{\beta_1}(\lambda) \in R(K_2), \quad \forall \lambda \in \widehat{K_1}.$$

We have the following extension of Theorem 2.9

Theorem 2.12. *We have $\mathcal{Q}_{K_1 \times K_2}^{\beta_1}(N) = \sum_{\lambda \in \widehat{K_1}} \theta^{\beta_1}(\lambda) \otimes V_{\lambda}^{K_1}$ where $\theta^{\beta_1}(\lambda) \neq 0$ only if $\|\lambda\| \geq \|\beta_1\|$.*

Proof. The proof works exactly like the one of Theorem 2.9. \square

Let us explain the “quantization commutes with reduction theorem”, or why we can consider the geometric quantization of

$$N//_0 K_1 := \Phi_1^{-1}(0)/K_1$$

as the K_1 -invariant part of $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(N)$.

Let us first suppose that 0 is a regular value of Φ_1 . Then $N//_0 K_1$ is a compact symplectic *orbifold* equipped with a Hamiltonian action of K_2 : the corresponding moment map is induced by the restriction of Φ_2 to $\Phi_1^{-1}(0)$. The symplectic quotient $N//_0 K_1$ is pre-quantized by the line orbibundle

$$L_0 := \left(L|_{\Phi_1^{-1}(0)} \right) / K_1.$$

Definition 1.1 extends to the orbifold case. We can still define the geometric quantization of $N//_0 K_1$ as the index of an elliptic operator: we denote it by $\mathcal{Q}_{K_2}(N//_0 K_1) \in R(K_2)$. We have

Theorem 2.13. *If 0 is a regular value of Φ_1 , the K_1 -invariant part of $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(N)$ is equal to $\mathcal{Q}_{K_2}(N//_0 K_1) \in R(K_2)$.*

Suppose now that 0 is not a regular value of Φ_1 . Let T_1 be a maximal torus of K_1 , and let $C_1 \subset \mathfrak{t}_1^*$ be a Weyl chamber. Since Φ_1 is proper, the convexity Theorem says that the image of Φ_1 intersects C_1 in a closed locally polyhedral convex set, that we denote $\Delta_{K_1}(N)$ [14].

We consider an element $a \in \Delta_{K_1}(N)$ which is generic and sufficiently close to $0 \in \Delta_{K_1}(N)$: we denote $(K_1)_a$ the subgroup of K_1 which stabilizes a . When $a \in \Delta_{K_1}(N)$ is generic, one can show (see [18]) that

$$N \parallel_a K_1 := \Phi_{K_1}^{-1}(a)/(K_1)_a$$

is a compact Hamiltonian K_2 -orbifold, and that

$$L_a := \left(L|_{\Phi_{K_1}^{-1}(a)} \right) / (K_1)_a.$$

is a K_2 -equivariant line orbibundle over $N \parallel_a K_1$: we can then define, like in Definition 1.1, the element $\mathcal{Q}_{K_2}(N \parallel_a K_1) \in R(K_2)$ as the equivariant index of the Dolbeault-Dirac operator on $N \parallel_a K_1$ (with coefficients in L_a).

Theorem 2.14. *The K_1 -invariant part of $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M)$ is equal to $\mathcal{Q}_{K_2}(N \parallel_a K_1) \in R(K_2)$. In particular, the elements $\mathcal{Q}_{K_2}(N \parallel_a K_1)$ do not depend on the choice of the generic element $a \in \Delta_H(N)$, when a is sufficiently close to 0.*

PROOFS OF THEOREM 2.13 AND THEOREM 2.14 . When N is compact and $K_2 = \{e\}$, the proofs can be found in [18] and in [19]. Let us explain briefly how the \mathbf{K} -theoretic proof of [19] extends naturally to our case. Like in Definition 2.10, we have the following decomposition

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(N) = \sum_{\beta \in \mathcal{B}_1} \mathcal{Q}_{K_1 \times K_2}^{\beta_1}(N),$$

And Theorem 2.12 tells us that $\left[\mathcal{Q}_{K_1 \times K_2}^{\beta_1}(N) \right]^{K_1} = 0$ if $\beta_1 \neq 0$. We have proved the first step:

$$\left[\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(N) \right]^{K_1} = \left[\mathcal{Q}_{K_1 \times K_2}^0(N) \right]^{K_1}.$$

The analysis of the term $\left[\mathcal{Q}_{K_1 \times K_2}^0(N) \right]^{K_1}$ is undertaken in [19] when $K_2 = \{e\}$: we explain that this term is equal either to $\mathcal{Q}(N \parallel_0 K_1)$ when 0 is a regular value, or to $\mathcal{Q}(N \parallel_a K_1)$ with a generic. It work similarly with an action of a compact Lie group K_2 . \square

Definition 2.15. *The geometric quantization of $N \parallel_0 K_1 := \Phi_1^{-1}(0)/K_1$ is taken as the K_1 -invariant part of $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(N)$. We denote it $\mathcal{Q}_{K_2}(N \parallel_0 K_1)$.*

2.4. Quantization of points. Let (M, Ω, Φ) be a proper Hamiltonian K -manifold prequantized by a Kostant-Souriau line bundle L . Let $\mu \in \hat{K}$ be dominant weight such that $\Phi^{-1}(K \cdot \mu)$ is a K -orbit in M . Let $m^o \in \Phi^{-1}(\mu)$ so that

$$\Phi^{-1}(K \cdot \mu) = K \cdot m^o$$

Then the reduced space $M_\mu := \Phi^{-1}(K \cdot \mu)/K$ is a point. The aim of this section is to compute the quantization of M_μ : $\mathcal{Q}(M_\mu) \in \mathbb{Z}$.

Let H be the stabilizer subgroup of m^o . We have a linear action of H on the 1-dimensional vector space $L_{m^o} \subset L$. We have $H \subset K_\mu$ where K_μ is the connected

subgroup of K that fixes $\mu \in \mathfrak{t}^*$. Let $\mathbb{C}_{-\mu}$ be the 1-dimensional representation of K_μ associated to the infinitesimal character $-i\mu$.

Let us denote χ be the character of H defined by the 1-dimensional representation $\mathbb{C}_\chi := L_{m^o} \otimes \mathbb{C}_{-\mu}$. We know from the Kostant formula (1.1) that $\chi = 1$ on the identity component $H^o \subset H$.

Theorem 2.16. *We have*

$$(2.22) \quad \mathcal{Q}(M_\mu) = \begin{cases} 1 & \text{if } \chi = 1 \text{ on } H \\ 0 & \text{in the other case.} \end{cases}$$

This Theorem tells us in particular that $\mathcal{Q}(M_\mu) = 1$ when the stabiliser subgroup $H \subset K$ of a point $m^o \in \Phi^{-1}(\mu)$ is *connected*.

Proof. Let $N = M \times \overline{K \cdot \mu}$ be the proper Hamiltonian K -manifold which is prequantized by the line bundle $L_N := L \otimes [\mathbb{C}_{-\mu}]$. Let us denote Φ_N the moment map on N . Since $\Phi^{-1}(K \cdot \mu)$ is a K -orbit in M , we see that $\Phi_N^{-1}(0)$ is the K -orbit through $n^o := (m^o, \mu)$ where $m^o \in \Phi^{-1}(\mu)$. Note that H is the stabilizer subgroups of n^o .

Let $\mathcal{Q}_K^{\Phi_N}(N) \in R^{-\infty}(K)$ be the formal quantization of N through the proper map Φ_N . By definition

$$\begin{aligned} \mathcal{Q}(M_\mu) &= \left[\mathcal{Q}_K^{\Phi_N}(N) \right]^K \\ &= \left[\mathcal{Q}_K^0(N) \right]^K. \end{aligned}$$

where $\mathcal{Q}_K^0(N)$ depends only of a neighborhood of $\Phi_N^{-1}(0)$.

The orbit $K \cdot n^o \hookrightarrow N$ is an isotropic embedding since it is the 0-level of the moment map Φ_N . Then to describe a K -invariant neighborhood of $K \cdot n^o$ in N we can use the normal-form recipe of Marle, Guillemin and Sternberg.

First we consider, following Weinstein (see [11, 31]), the symplectic normal bundle

$$(2.23) \quad \mathcal{V} := T(K \cdot n^o)^{\perp, \Omega} / T(K \cdot n^o),$$

where the orthogonal $(^{\perp, \Omega})$ is taken relatively to the symplectic 2-form. We have

$$\mathcal{V} = K \times_H V$$

where the vector space $V := T_{n^o}(K \cdot n^o)^{\perp, \Omega} / T_{n^o}(K \cdot n^o)$ inherits a symplectic structure and an Hamiltonian action of the group H : we denote $\Phi_H : V \rightarrow \mathfrak{h}^*$ the corresponding moment map.

Consider now the following symplectic manifold

$$(2.24) \quad \tilde{N} := \mathcal{V} \oplus T^*(K/H) = K \times_H \left((\mathfrak{k}/\mathfrak{h})^* \oplus V \right).$$

The action of H on \tilde{N} is Hamiltonian and the moment map $\Phi_{\tilde{N}} : \tilde{N} \rightarrow \mathfrak{k}^*$ is given by the equation

$$(2.25) \quad \Phi_{\tilde{N}}([k; \xi, v]) = k \cdot (\xi + \Phi_H(v)) \quad k \in K, \xi \in (\mathfrak{k}/\mathfrak{h})^*, v \in V.$$

The Hamiltonian K -manifold \tilde{N} is prequantized by the line bundle $L_{\tilde{N}} := K \times_H \mathbb{C}_\chi$.

The *local normal form* Theorem (see [10], [24] Proposition 2.5) tells us that there exists a K -Hamiltonian isomorphism $\Upsilon : \mathcal{U}_1 \xrightarrow{\sim} \mathcal{U}_2$ between a K -invariant neighborhood \mathcal{U}_1 of $K \cdot n^o$ in N , and a K -invariant neighborhood \mathcal{U}_2 of K/H in \tilde{N} . This isomorphism Υ , when restricted to $K \cdot n^o$, corresponds to the natural isomorphism $K \cdot n^o \xrightarrow{\sim} K/H$.

Thanks to Υ , we know that the fiber $\Phi_H^{-1}(0) \subset V$ is reduced to $\{0\}$. This last point is equivalent to the fact that Φ_H (and then $\Phi_{\tilde{N}}$) is proper map (see [21]). We check easily that the set of critical points of $\|\Phi_{\tilde{N}}\|^2$ is reduced to $\Phi_{\tilde{N}}^{-1}(0) = K/H$. Then, thank to the isomorphism Υ , we have that

$$(2.26) \quad \mathcal{Q}_K^0(N) = \mathcal{Q}_K^0(\tilde{N}) = \mathcal{Q}_K^{\Phi_{\tilde{N}}}(\tilde{N}).$$

Let $\text{Ind}_H^K : R^{-\infty}(H) \rightarrow R^{-\infty}(K)$ be the induction map that is defined by the relation $\langle \text{Ind}_H^K(\phi), E \rangle = \langle \phi, E|_H \rangle$ for any $\phi \in R^{-\infty}(H)$ and $E \in R(K)$. Note that

$$[\text{Ind}_H^K(\phi)]^K = \langle \text{Ind}_H^K(\phi), \mathbb{C} \rangle = \langle \phi, \mathbb{C} \rangle = [\phi]^H.$$

Since $\Phi_H : V \rightarrow \mathfrak{h}^*$ is proper one can consider the quantization of the vector space V through the map $\Phi_H : \mathcal{Q}_H^{\Phi_H}(V) \in R^{-\infty}(H)$.

Proposition 2.17. • *We have*

$$(2.27) \quad \mathcal{Q}_K^{\Phi_{\tilde{N}}}(\tilde{N}) = \text{Ind}_H^K \left(\mathcal{Q}_H^{\Phi_H}(V) \otimes \mathbb{C}_\chi \right)$$

- *The formal quantization $\mathcal{Q}_H^{\Phi_H}(V)$ coincides, as a generalized H -module, to the H -module $S(V^*)$ of polynomial function on V .*
- *The set $[S(V^*)]^{H^o}$ of polynomials invariant by the connected component H^o is reduced to the scalars.*

With the last Proposition we can finish the proof of Theorem 2.16 as follows. We have

$$\begin{aligned} \mathcal{Q}(M_\mu) &= [\mathcal{Q}_K^{\Phi}(N)]^K \\ &= [\mathcal{Q}_K^{\Phi_{\tilde{N}}}(\tilde{N})]^K \\ &= [\mathcal{Q}_H^{\Phi_H}(V) \otimes \mathbb{C}_\chi]^H \\ &= [S(V^*) \otimes \mathbb{C}_\chi]^H = [\mathbb{C}_\chi]^H. \end{aligned}$$

Proof. The first point of Proposition 2.17 follows from the property of induction defined by Atiyah (see Section 3.4 in [19]). Let us explain the arguments. We work with the H -manifold $\mathcal{Y} = (\mathfrak{k}/\mathfrak{h})^* \oplus V$ and the H -equivariant map $j : \mathcal{Y} \hookrightarrow \tilde{N} := K \times_H \mathcal{Y}, y \mapsto [e, y]$.

We notice⁵ that $T\tilde{N} \simeq K \times_H (\mathfrak{k}/\mathfrak{h} \oplus T\mathcal{Y})$, and that $T_K \tilde{N} \simeq K \times_H (T_H \mathcal{Y})$. Hence the map j induces an isomorphism $j_* : K_H(T_H \mathcal{Y}) \rightarrow K_K(T_K \tilde{N})$. Theorem 4.1 of

⁵ These identities come from the following $K \times H$ -equivariant isomorphism of vector bundles over $K \times \mathcal{Y}$: $T_H(\tilde{N}) \rightarrow K \times (\mathfrak{k}/\mathfrak{h} \oplus T\mathcal{Y}), (k, m; \frac{d}{dt}|_{t=0}(ke^{tX}) + v_m) \mapsto (k, m; pr_{\mathfrak{k}/\mathfrak{h}}(X) + v_m)$. Here $pr_{\mathfrak{k}/\mathfrak{h}} : \mathfrak{k} \rightarrow \mathfrak{k}/\mathfrak{h}$ is the orthogonal projection.

Atiyah [1] tells us that the following diagram

$$(2.28) \quad \begin{array}{ccc} K_H(T_H \mathcal{Y}) & \xrightarrow{j_*} & K_K(T_K \tilde{N}) \\ \text{Index}_Y^H \downarrow & & \downarrow \text{Index}_N^K \\ R^{-\infty}(H) & \xrightarrow{\text{Ind}_H^K} & R^{-\infty}(K) . \end{array}$$

is commutative.

The tangent bundle $T\tilde{N}$ is equivariantly diffeomorphic to

$$K \times_H [\mathfrak{k}/\mathfrak{h} \oplus (\mathfrak{k}/\mathfrak{h})^* \oplus TV] \simeq K \times_H [(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \oplus TV]$$

where $(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}}$ is the complexification of the real vector space $\mathfrak{k}/\mathfrak{h}$. We consider on \tilde{N} the almost complex structure $J_{\tilde{N}} = (i, J_V)$ where i is the complex structure on $(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}}$ and J_V is a compatible (constant) complex structure on the symplectic vector space V . Note that $J_{\tilde{N}}$ is compatible with the symplectic structure on a neighborhood U of the 0-section of the bundle $\tilde{N} \rightarrow K/H$.

Let $\kappa_{\tilde{N}}$ be the Kirwan vector field on \tilde{N} :

$$\kappa_{\tilde{N}}([k; \xi, v]) = -\xi + i[\xi, \Phi_H(v)] \oplus \kappa_V(v) \in (\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \oplus V.$$

Here κ_V is the Kirwan vector field relative to the Hamiltonian action of H on the symplectic vector space V . Note that $\kappa_{\tilde{N}}$ vanishes exactly on the 0-section of the bundle $\tilde{N} \rightarrow K/H$.

Let $\mathbf{c}^{\kappa_{\tilde{N}}}$ be the symbol $\text{Thom}(\tilde{N}, J_{\tilde{N}}) \otimes L_{\tilde{N}}$ pushed by the vector field $\kappa_{\tilde{N}}$. The generalized character $\mathcal{Q}_K^{\Phi_{\tilde{N}}}(\tilde{N})$ is either computed as the equivariant index of the symbols $\mathbf{c}^{\kappa_{\tilde{N}}}$ or $\mathbf{c}^{\kappa_{\tilde{N}}}|_U$.

Remark 2.18. *The fact that $J_{\tilde{N}}$ is not compatible on the entire manifold \tilde{N} is not problematic, since $J_{\tilde{N}}$ is compatible in a neighborhood U of the set where $\kappa_{\tilde{N}}$ vanishes. See the first point of Lemma 2.6.*

For $X + i\eta \oplus w \in T_{[k; \xi, v]}\tilde{N} \simeq (\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \oplus V$, the map

$$(2.29) \quad \mathbf{c}^{\kappa_{\tilde{N}}}(X + i\eta \oplus w) = \mathbf{c}\left(X + \xi + i(\eta - [\xi, \Phi_H(v)])\right) \odot \mathbf{c}\left(w - \kappa_V(v)\right)$$

acts on the vector space $\wedge_{\mathbb{C}}(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \otimes \wedge_{J_V} V \otimes \mathbb{C}_{\chi}$.

Let $\text{Bott}(\mathfrak{k}/\mathfrak{h})$ be the Bott morphism of the vector space $\mathfrak{k}/\mathfrak{h}$. It is an elliptic morphism defined by

$$\text{Bott}(\mathfrak{k}/\mathfrak{h})|_{\xi}(\eta) = \mathbf{c}(\xi + i\eta) \text{ acting on } \wedge_{\mathbb{C}}(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}},$$

for $\eta \in T_{\xi}(\mathfrak{k}/\mathfrak{h})$. Let \mathbf{c}^{κ_V} be the symbol $\text{Thom}(V, J_V)$ pushed by the vector field κ_V .

Lemma 2.19. *We have*

$$\mathbf{c}^{\kappa_{\tilde{N}}} = j_*\left(\text{Bott}(\mathfrak{k}/\mathfrak{h}) \odot \mathbf{c}^{\kappa_V} \otimes \mathbb{C}_{\chi}\right).$$

Proof. We work with the symbol

$$\sigma^T|_{(\xi, v)}(\eta) = \mathbf{c}(\xi + i\eta - iT[\xi, \Phi_H(v)])$$

acting on $\wedge_{\mathbb{C}}(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}}$. Note that $\text{Bott}(\mathfrak{k}/\mathfrak{h}) = \sigma^0$. From (2.29), we see that $\mathbf{c}^{\kappa_{\tilde{N}}} = j_*\left(\sigma^1 \odot \mathbf{c}^{\kappa_V} \otimes \mathbb{C}_{\chi}\right)$. It is now easy to check that $\sigma^T \odot \mathbf{c}^{\kappa_V} \otimes \mathbb{C}_{\chi}$, $T \in [0, 1]$ is an homotopy of transversally elliptic symbols on $\mathfrak{k}/\mathfrak{h} \times V$. \square

The commutative diagram (2.28) and the last Lemma gives

$$\begin{aligned}
\mathcal{Q}_K^{\Phi_{\tilde{N}}}(\tilde{N}) &= \text{Index}_N^K(\mathbf{c}^{\kappa_{\tilde{N}}}) \\
&= \text{Ind}_H^K \left(\text{Index}_{\mathfrak{t}/\mathfrak{h} \times V}^K \left(\text{Bott}(\mathfrak{t}/\mathfrak{h}) \odot \mathbf{c}^{\kappa_V} \right) \otimes \mathbb{C}_\chi \right) \\
&= \text{Ind}_H^K \left(\text{Index}_{\mathfrak{t}/\mathfrak{h}}^K(\text{Bott}(\mathfrak{t}/\mathfrak{h})) \otimes \text{Index}_V^K(\mathbf{c}^{\kappa_V}) \otimes \mathbb{C}_\chi \right) \\
&= \text{Ind}_H^K \left(\mathcal{Q}_H^{\Phi_H}(V) \otimes \mathbb{C}_\chi \right).
\end{aligned}$$

We have used here that the equivariant index of $\text{Bott}(\mathfrak{t}/\mathfrak{h})$ is equal to 1 (e.g. the trivial representation).

Let us prove now the second point of Proposition 2.17. The Kirwan vector field κ^V satisfies the simple rule:

$$(2.30) \quad (\kappa^V(v), J_V v) = -\Omega(\kappa^V(v), v) = \frac{1}{2} \|\Phi_H(v)\|^2, \quad v \in V.$$

It shows in particular that $\kappa^V(v) = 0 \Leftrightarrow \Phi_H(v) = 0$. Since the moment map $\Phi_H : V \rightarrow \mathfrak{h}^*$ is quadratic, the fact that Φ_H is proper is equivalent to the fact that $\Phi_H^{-1}(0) = 0$.

We consider on V the family of symbol σ^s :

$$\sigma^s|_v(w) = \mathbf{c}(w - s\kappa^V(v) - (1-s)J_V v)$$

viewed as a map from $\wedge_{\mathbb{C}}^{\text{even}} V$ to $\wedge_{\mathbb{C}}^{\text{odd}} V$. Thanks to (2.30), one sees that σ^s is a family of K -transversally elliptic symbol on V . Hence $\sigma^1 = \mathbf{c}^{\kappa_V}$ and $\sigma^0 = \mathbf{c}(w - J_V v)$ defines the same class in the group $K_K(\text{TK}V)$. The symbol σ^0 was first studied by Atiyah [1] when $\dim_{\mathbb{C}} V = 1$. The author considered the general case in [19]. We have

$$\text{Index}_V^K(\sigma^0) = S(V^*) \quad \text{in } R^{-\infty}(K).$$

The last point of Proposition 2.17 is a consequence of the properness of the moment map Φ_H (see Section 5 of [21]).

□

□

Example 2.20 ([21]). *We consider the action of the unitary group U_n on \mathbb{C}^n . The symplectic form on \mathbb{C}^n is defined by $\Omega(v, w) = \frac{i}{2} \sum_k v_k \overline{w_k} - \overline{v_k} w_k$. Let us identify the Lie algebra \mathfrak{u}_n with its dual through the trace map. The moment map $\Phi : \mathbb{C}^n \rightarrow \mathfrak{u}_n$ is defined by $\Phi(v) = \frac{1}{2i} v \otimes v^*$ where $v \otimes v^* : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is the linear map $w \mapsto (\sum_k \overline{v_k} w_k) v$. One checks easily that the pull-back by Φ of a U_n -orbit in \mathfrak{u}_n is either empty or a U_n -orbit in \mathbb{C}^n . We know also that the stabiliser subgroup of a non-zero vector of \mathbb{C}^n is connected since it is diffeomorphic to U_{n-1} . Finally we have*

$$(2.31) \quad \mathcal{Q}((\mathbb{C}^n)_\mu) = \begin{cases} 1 & \text{if } \mu \in \widehat{U_n} \text{ belongs to the image of } \Phi \\ 0 & \text{if } \mu \in \widehat{U_n} \text{ does not belong to the image of } \Phi. \end{cases}$$

Then one checks that $\mathcal{Q}_{U_n}^{-\infty}(\mathbb{C}^n)$ coincides in $R^{-\infty}(U_n)$ with the algebra $S((\mathbb{C}^n)^)$ of polynomial function on \mathbb{C}^n .*

Example 2.21 ([23]). We consider the Lie group $\mathrm{SL}_2(\mathbb{R})$ and its compact torus of dimension 1 denoted by T . The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is identified with its dual through the trace map, and the Lie algebra \mathfrak{t} is naturally identified with $\mathfrak{sl}_2(\mathbb{R})^T$. For $l \in \mathbb{Z} \setminus \{0\}$, we consider the character χ_l of T defined by

$$\chi_l \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = e^{il\theta}.$$

Its differential $\frac{1}{i}d\chi_l \in \mathfrak{t}^*$ correspond (through the trace map) to the matrix

$$X_l = \begin{pmatrix} 0 & l/2 \\ -l/2 & 0 \end{pmatrix}.$$

Let \mathcal{O}_l be the coadjoint orbit of the group $\mathrm{SL}_2(\mathbb{R})$ through the matrix X_l . It is a Hamiltonian $\mathrm{SL}_2(\mathbb{R})$ -manifold prequantized by the $\mathrm{SL}_2(\mathbb{R})$ -equivariant line bundle $L_l \simeq \mathrm{SL}_2(\mathbb{R}) \times_T \mathbb{C}_l$, where \mathbb{C}_l is the T -module associated to the character χ_l . We look at the Hamiltonian action of T on \mathcal{O}_l . Let $\Phi_T : \mathcal{O}_l \rightarrow \mathfrak{t}^*$ be the corresponding moment map. One checks that the moment map Φ_T is proper and that its image is equal to the half-line $\{aX_l, a \geq 1\} \subset \mathfrak{t}^*$.

We check that for each $\xi \in \{aX_l, a \geq 1\}$ the fiber $\Phi_T^{-1}(\xi)$ is equal to a T -orbit in \mathcal{O}_l . For $k \in \mathbb{Z}$, let us denote $(\mathcal{O}_l)_k$ the symplectic reduction of \mathcal{O}_l at the level X_k . We know that $(\mathcal{O}_l)_k = \emptyset$ if $k \notin \{al, a \geq 1\}$, and that $(\mathcal{O}_l)_k$ is a point if $k \in \{al, a \geq 1\}$.

In order to compute $\mathcal{Q}((\mathcal{O}_l)_k)$ we look at the stabilizer subgroup $T_m := \{t \in T \mid t \cdot m = m\}$ for each point $m \in \mathcal{O}_l$. One sees that $T_m = T$ if $m = X_l$ and T_m is equal to the center $\{\pm Id\}$ of $\mathrm{SL}_2(\mathbb{R})$, when $m \neq X_l$.

Theorem 2.16 gives in this setting that, for $k \in \{al, a \geq 1\}$,

$$(2.32) \quad \mathcal{Q}((\mathcal{O}_l)_k) = \begin{cases} 1 & \text{if } l - k \text{ is even} \\ 0 & \text{if } l - k \text{ is odd.} \end{cases}$$

Hence the formal geometric quantization of the proper T -manifold \mathcal{O}_l is

$$(2.33) \quad \mathcal{Q}_T^{-\infty}(\mathcal{O}_l) = \begin{cases} \mathbb{C}_l \cdot \sum_{p \geq 0} \mathbb{C}_{2p} & \text{if } l > 0 \\ \mathbb{C}_l \cdot \sum_{p \geq 0} \mathbb{C}_{-2p} & \text{if } l < 0. \end{cases}$$

Here we recognize that $\mathcal{Q}_T^{-\infty}(\mathcal{O}_l)$ coincides with the restriction of the holomorphic (resp. anti-holomorphic) discrete series representation Θ_l to the group T when $l > 0$ (resp. $l < 0$).

2.5. Wonderful compactifications and symplectic cuts. Another equivalent definition of the quantization $\mathcal{Q}^{-\infty}$ uses a generalisation of the technique of symplectic cutting (originally due to Lerman [13]) that was introduced in [21] and was motivated by the wonderful compactifications of De Concini and Procesi. Let us recall the method.

We recall that T is a maximal torus in the compact connected Lie group K , and W is the Weyl group. We define a K -adapted polytope in \mathfrak{t}^* to be a W -invariant Delzant polytope P in \mathfrak{t}^* whose vertices are regular elements of the weight lattice Λ^* . If $\{\lambda_1, \dots, \lambda_N\}$ are the dominant weights lying in the union of all the closed one-dimensional faces of P , then there is a $G \times G$ -equivariant embedding of $G = K_{\mathbb{C}}$ into

$$\mathbb{P}\left(\bigoplus_{i=1}^N V_{\lambda_i}^* \otimes V_{\lambda_i}\right)$$

associating to $g \in G$ its representation on $\bigoplus_{i=1}^N V_{\lambda_i}$. The closure \mathcal{X}_P of the image of G in this projective space is smooth and is equipped with a $K \times K$ that we denote:

$$(k_1, k_2) \cdot x = k_2 \cdot x \cdot k_1^{-1}.$$

Let $\Omega_{\mathcal{X}_P}$ be the symplectic 2-form on \mathcal{X}_P which given by the Kahler structure. We recall briefly the different properties of $(\mathcal{X}_P, \Omega_{\mathcal{X}_P})$: all the details can be found in [21].

- (1) \mathcal{X}_P is equipped with an Hamiltonian action of $K \times K$. Let $\Phi = (\Phi_l, \Phi_r) : M \rightarrow \mathfrak{k}^* \times \mathfrak{k}^*$ be the corresponding moment map.
- (2) The image of $\Phi := (\Phi_l, \Phi_r)$ is equal to $\{(k \cdot \xi, -k' \cdot \xi) \mid \xi \in P \text{ and } k, k' \in K\}$.
- (3) The Hamiltonian manifold $(\mathcal{X}_P, K \times K)$ has no multiplicities: the pull-back by Φ of a $K \times K$ -orbit in the image is a $K \times K$ -orbit in \mathcal{X}_P .

Let $\mathcal{U}_P := K \cdot P^\circ$ where P° is the interior of P . We define

$$\mathcal{X}_P^\circ := \Phi_l^{-1}(\mathcal{U}_P)$$

which is an invariant, open and dense subset of \mathcal{X}_P . We have the following important property concerning \mathcal{X}_P° .

- (4) There exists an equivariant diffeomorphism $\Upsilon : K \times \mathcal{U}_P \rightarrow \mathcal{X}_P^\circ$ such that $\Upsilon^*(\Phi_l)(k, \xi) = k \cdot \xi$ and $\Upsilon^*(\Phi_r)(k, \xi) = -\xi$.
- (5) This diffeomorphism Υ is a quasi-symplectomorphism in the sense that there is a homotopy of symplectic forms taking the symplectic form on the open subset $K \times \mathcal{U}_P$ of the cotangent bundle T^*K to the pullback of the symplectic form $\Omega_{\mathcal{X}_P}$ on \mathcal{X}_P° .
- (6) The symplectic manifold $(\mathcal{X}_P, \Omega_{\mathcal{X}_P})$ is prequantized by the restriction of the hyperplane line bundle $\mathcal{O}(1) \rightarrow \mathbb{P}(\bigoplus_{i=1}^N V_{\lambda_i}^* \otimes V_{\lambda_i})$ to \mathcal{X}_P : let us denote L_P the corresponding $K \times K$ -equivariant line bundle.
- (7) The pull-back of the line bundle L_P by the map $\Upsilon : K \times \mathcal{U}_P \hookrightarrow \mathcal{X}_P$ is trivial.

Let (M, Ω_M, Φ_M) be a proper Hamiltonian K -manifold. We also consider the Hamiltonian $K \times K$ -manifold \mathcal{X}_P associated to a K -adapted polytope P . We consider now the product $M \times \mathcal{X}_P$ with the following $K \times K$ action:

- the action $k \cdot_1 (m, x) = (k \cdot m, x \cdot k^{-1})$: the corresponding moment map is $\Phi_1(m, x) = \Phi_M(m) + \Phi_r(x)$,
- the action $k \cdot_2 (m, x) = (m, k \cdot x)$: the corresponding moment map is $\Phi_2(m, x) = \Phi_l(x)$.

Definition 2.22. We denote M_P the symplectic reduction at 0 of $M \times \mathcal{X}_P$ for the action $\cdot_1 : M_P := (\Phi_1)^{-1}(0)/(K, \cdot_1)$.

Then M_P inherits a Hamiltonian K -action with moment map $\Phi_{M_P} : M_P \rightarrow \mathfrak{k}^*$ whose image is $\Phi(M) \cap K \cdot P$.

One checks that M_P contains an open and dense subset of smooth points which quasi-symplectomorphic to the open subset $(\Phi_M)^{-1}(\mathcal{U}_P)$. If the polytope P is fixed, we can work with the dilated polytopes nP for $n \geq 1$. We have then the family of compact, perhaps singular, K -hamiltonian manifolds M_{nP} , $n \geq 1$: in Section 2.3, we have explained how was defined their geometric quantization $\mathcal{Q}_K(M_{nP}) \in R(K)$.

We have a convenient definition for $\mathcal{Q}^{-\infty}$.

Proposition 2.23 ([21]). *We have the following equality in $R^{-\infty}(K)$:*

$$(2.34) \quad \mathcal{Q}_K^{-\infty}(M) = \lim_{n \rightarrow \infty} \mathcal{Q}_K(M_{nP}).$$

3. PROOF OF THEOREM 1.4

The main result of this section is

Theorem 3.1. *Let $r_P := \inf_{\xi \in \partial P} \|\xi\|$. The generalized character*

$$\mathcal{Q}_K^\Phi(M) - \mathcal{Q}_K(M_P) \in R^{-\infty}(K)$$

is supported outside the ball B_{r_P} .

Then, for the dilated polytope $nP, n \geq 1$, the character $\mathcal{Q}_K^\Phi(M) - \mathcal{Q}_K(M_{nP})$ is supported outside the ball B_{nr_P} . Taking the limit when n goes to infinity gives

$$(3.35) \quad \mathcal{Q}_K^\Phi(M) = \lim_{n \rightarrow \infty} \mathcal{Q}_K(M_{nP}).$$

Finally, the identity of Theorem 1.4,

$$\mathcal{Q}_K^\Phi(M) = \mathcal{Q}_K^{-\infty}(M),$$

is a direct consequence of (2.34) and (3.35).

Recall that $O(r) \in R^{-\infty}(K)$ denoted any generalized character supported outside the ball B_r .

Theorem 3.1 follows from the comparison of three different geometrical situation. All of them concern Hamiltonian actions of $K_1 \times K_2$, where K_1 and K_2 are two copies of K .

First setting. We work with the Hamiltonian $K_1 \times K_2$ -manifold $M \times \mathcal{X}_P$: here K_1 acts both on M and on \mathcal{X}_P . Since the moment map Φ_1 (relative to the K_1 -action) is proper we may “quantize” $M \times \mathcal{X}_P$ via the map $\|\Phi_1\|^2$: let

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M \times \mathcal{X}_P) \in R^{-\infty}(K_1 \times K_2)$$

be the corresponding generalized character. Recall that $\mathcal{Q}_{K_2}(M_P)$ is equal to $[\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M \times \mathcal{X}_P)]^{K_1}$.

Second setting. We consider the same setting than before : the Hamiltonian action of $K_1 \times K_2$ on $M \times \mathcal{X}_P$. But we “quantize” $M \times \mathcal{X}_P$ through the global moment map $\Phi = (\Phi_1, \Phi_2)$. Here we have some liberty in the choice of the scalar product on $\mathfrak{k}_1^* \times \mathfrak{k}_2^*$. If $\|\xi\|^2$ is an invariant Euclidean norm on \mathfrak{k}^* , we take on $\mathfrak{k}_1^* \times \mathfrak{k}_2^*$ the Euclidean norm

$$(3.36) \quad \|(\xi_1, \xi_2)\|_\rho^2 = \|\xi_1\|^2 + \rho \|\xi_2\|^2$$

depending on a parameter $\rho > 0$. Let us consider the quantization of $M \times \mathcal{X}_P$ via the map $\|\Phi\|_\rho^2$:

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) \in R^{-\infty}(K_1 \times K_2).$$

Third setting. We consider the cotangent bundle T^*K with the Hamiltonian action of $K_1 \times K_2$: K_1 acts by *right* translations, and K_2 by *left* translations. We consider the Hamiltonian action of $K_1 \times K_2$ on $M \times T^*K$: here K_1 acts both on

M and on T^*K . Let $\Phi = (\Phi_1, \Phi_2)$ be the global moment map on $M \times T^*K$. Since the moment map Φ is proper we can “quantize” $M \times T^*K$ via the map $\|\Phi\|_\rho^2$: let

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K) \in R^{-\infty}(K_1 \times K_2)$$

be the corresponding generalized character.

Theorem 3.1 is a consequence of the following propositions.

First we compare $\mathcal{Q}_{K_2}^\Phi(M)$ with the K_1 -invariant part of $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K)$.

Proposition 3.2. *For any $\rho \in]0, 1]$, we have*

$$(3.37) \quad \left[\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K) \right]^{K_1} = \mathcal{Q}_{K_2}^\Phi(M) \quad \text{in} \quad R^{-\infty}(K_2).$$

Then we compare the K_1 -invariant part of the generalized characters $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K)$ and $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P)$.

Proposition 3.3. *For any $\rho \in]0, 1]$, we have the following relation in $R^{-\infty}(K_2)$*

$$(3.38) \quad \left[\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) \right]^{K_1} - \left[\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K) \right]^{K_1} = O(r_P)$$

Finally we compare the K_1 -invariant part of the generalized characters $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P)$ and $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M \times \mathcal{X}_P)$.

Proposition 3.4. *There exists $\epsilon > 0$ such that*

$$(3.39) \quad \mathcal{Q}_{K_2}(M_P) - \left[\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) \right]^{K_1} = O((\epsilon/\rho)^{1/2}) \quad \text{in} \quad R^{-\infty}(K_2)$$

if $\rho > 0$ is small enough.

If we sum the relations (3.37), (3.38) and (3.39) we get

$$\mathcal{Q}_{K_2}^\Phi(M) = \mathcal{Q}_{K_2}(M_P) + O(r_P) + O((\epsilon/\rho)^{1/2})$$

if ρ is small enough. So Theorem 3.1 follows by taking $(\epsilon/\rho)^{1/2} \geq r_P$.

3.1. Proof of Proposition 3.2. The cotangent bundle T^*K is identified with $K \times \mathfrak{k}^*$. The data is then (see Section 5.1):

- the Liouville 1-form $\lambda = \sum_j \omega_j \otimes E_j$. Here (E_j) is a basis of \mathfrak{k} with dual basis (E_j^*) , and ω_j is the left invariant 1-form on K defined by $\omega_j(\frac{d}{dt}a e^{tX}|_0) = \langle E_j^*, X \rangle$.
- the symplectic form $\Omega := -d\lambda$,
- the action of $K_1 \times K_2$ on $K \times \mathfrak{k}^*$ is $(k_1, k_2) \cdot (a, \xi) = (k_2 a k_1^{-1}, k_1 \cdot \xi)$,
- the moment map relative to the K_1 -action is $\Phi_r(a, \xi) = -\xi$,
- the moment map relative to the K_2 -action is $\Phi_l(a, \xi) = a \cdot \xi$.

We work now with the Hamiltonian action of $K_1 \times K_2$ on $M \times T^*K$ given by

$$(k_1, k_2) \cdot (m, a, \xi) = (k_1 \cdot m, k_2 a k_1^{-1}, k_1 \cdot \xi).$$

The corresponding moment map is $\Phi = (\Phi_1, \Phi_2)$: $\Phi_1(m, a, \xi) = \Phi_M(m) - \xi$ and $\Phi_2(m, a, \xi) = a \cdot \xi$.

Let \mathbf{c}_1 be a symbol $\text{Thom}(M, J_1) \otimes L$ attached to the prequantized Hamiltonian K_1 -manifold (M, Ω) . The cotangent bundle T^*K is prequantized by the trivial line bundle: let \mathbf{c}_2 be the symbol $\text{Thom}(T^*K, J_2)$ attached to the prequantized

Hamiltonian $K_1 \times K_2$ -manifold T^*K . The product $\mathbf{c} = \mathbf{c}_1 \odot \mathbf{c}_2$ corresponds to the symbol $\text{Thom}(N, J) \otimes L$ on $N = M \times T^*K$.

Let κ_ρ be the Kirwan vector field associated to the map $\|\Phi\|_\rho^2 : M \times T^*K \rightarrow \mathbb{R}$. We check that $\|\Phi\|_\rho^2(m, k, \xi) = \|\Phi_M(m) - \xi\|^2 + \rho\|\xi\|^2$, and

$$\kappa_\rho(m, k, \xi) = \left(\underbrace{(\Phi_M(m) - \xi) \cdot m}_{\kappa_I}; \underbrace{\tilde{\Phi}_M(m) - (1 + \rho)\tilde{\xi}}_{\kappa_{II, \rho}}; \underbrace{-[\tilde{\Phi}_M(m), \tilde{\xi}]}_{\kappa_{III}} \right).$$

Here $T_{(m, k, \xi)}(M \times T^*K) \simeq T_m M \times \mathfrak{k} \times \mathfrak{k}$. We have

$$\begin{aligned} \text{Cr}(\|\Phi\|_\rho^2) &= \{\kappa_\rho = 0\} \\ &= \bigcup_{\beta \in \mathcal{B}} K_1 \times K_2 \cdot \left[M^{\tilde{\beta}} \cap \Phi_M^{-1}(\beta) \times \{1\} \times \left\{ \frac{\beta}{\rho + 1} \right\} \right] \end{aligned}$$

where \mathcal{B} parametrizes $\text{Cr}(\|\Phi_M\|^2)$. Hence one checks that the critical values of $\|\Phi\|_\rho^2$ are $\frac{\rho}{\rho+1}\|\beta\|^2, \beta \in \mathcal{B}$.

Let \mathbf{c}^{κ_ρ} be the symbol \mathbf{c} pushed by the vector field κ_ρ : we have

$$\mathbf{c}^{\kappa_\rho}(v; X; Y) = \mathbf{c}_1(v - \kappa_I) \odot \mathbf{c}_2(X - \kappa_{II, \rho}; Y - \kappa_{III})$$

for $(v; X; Y) \in T_{(m, k, \xi)}(M \times T^*K) \simeq T_m M \times \mathfrak{k} \times \mathfrak{k}$.

For a real $R > 0$ we define the open invariant subsets of $M \times T^*K$

$$\begin{aligned} U_R &:= \{\|\Phi\|_\rho^2 < R\} \\ V_R &:= \{\|\Phi_M\|^2 < R\} \times T^*K. \end{aligned}$$

By definition the generalized index $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K)$ is defined as the limit of the equivariant index

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(U_R) := \text{Index}_{\mathcal{U}_R}^{K_1 \times K_2}(\mathbf{c}^{\kappa_\rho}|_{U_R}),$$

when R goes to infinity (and stays outside the critical values of $\|\Phi\|_\rho^2$).

In the other hand, when R' is a regular value of $\|\Phi_M\|^2$, we see that the symbol $\mathbf{c}_\rho|_{V_{R'}}$ is $K_1 \times K_2$ -transversally elliptic. Let

$$(3.40) \quad \text{Index}_{V_{R'}}^{K_1 \times K_2}(\mathbf{c}^{\kappa_\rho}|_{V_{R'}})$$

be its equivariant index. Notice that the index map is well-defined on $V_R = \{\|\Phi_M\|^2 < R\} \times T^*K$ since T^*K can be seen as a open subset of a compact manifold.

It is easy to check that for any $R > 0$ there exists $R' > R$ such that $U_R \subset V_{R'}$. It implies that $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K)$ is also defined as the limit of (3.40) when R' goes to infinity.

We look now to the deformation $\kappa_\rho(s) = (\kappa_I^s; \kappa_{II, \rho}^s; s\kappa_{III})$, $s \in [0, 1]$ where

$$\kappa_I^s(m, \xi) = (\Phi_M(m) - s\xi) \cdot m \quad \text{and} \quad \kappa_{II, \rho}^s(m, \xi) = s\tilde{\Phi}_M(m) - (1 + s\rho)\tilde{\xi}.$$

Let $\mathbf{c}^{\kappa_\rho(s)}$ be the symbol \mathbf{c} pushed by the vector field $\kappa_\rho(s)$.

Lemma 3.5. *Let R' be a regular value of $\|\Phi_M\|^2$.*

- *The family $\mathbf{c}^{\kappa_\rho(s)}|_{V_{R'}}$, $s \in [0, 1]$ defines an homotopy of $K_1 \times K_2$ -transversally elliptic symbols on $V_{R'}$.*
- *The K_1 -invariant part of $\text{Index}_{V_{R'}}^{K_1 \times K_2}(\mathbf{c}^{\kappa_\rho(0)}|_{V_{R'}})$ is equal to $\mathcal{Q}_{K_2}^\Phi(M_{<R'})$.*

Proof. The first point follows from the fact that $\text{Char}(\mathbf{c}^{\kappa_\rho(s)}|_{V_{R'}}) \cap \text{T}_{K_1 \times K_2}(V_{R'})$, which is equal to

$$\left\{ (m, k, \frac{s}{1+s\rho} \Phi_M(m)), \ k \in K \text{ and } m \in \text{Cr}(\|\Phi_M\|^2) \cap \{\|\Phi_M\|^2 < R'\} \right\},$$

stays in a compact set when $s \in [0, 1]$.

The symbol $\mathbf{c}^{\kappa_\rho(0)}|_{V_{R'}}$ is equal to the product of the symbol $\mathbf{c}_1^\kappa|_{M < R'}$, which is K_1 -transversally elliptic, with the symbol

$$\mathbf{c}_2^\kappa(X; Y) = \mathbf{c}_2(X + \xi; Y)$$

which is a K_2 -transversally elliptic on T^*K . A basic computation done in section 5.1.2 gives that

$$\begin{aligned} \text{Index}_{\text{T}^*K}^{K_1 \times K_2}(\mathbf{c}_2^\kappa) &= \text{L}^2(K) \\ &= \sum_{\mu \in \widehat{K}} (V_\mu^{K_1})^* \otimes V_\mu^{K_2} \end{aligned}$$

in $R^{-\infty}(K_1 \times K_2)$. Finally the “multiplicative property” (see Theorem 2.1) gives

$$\begin{aligned} \text{Index}_{V_{R'}}^{K_1 \times K_2}(\mathbf{c}^{\kappa_\rho(0)}|_{V_{R'}}) &= \text{Index}_{M < R'}^{K_1}(\mathbf{c}_1^\kappa|_{M < R'}) \otimes \text{Index}_{\text{T}^*K}^{K_1 \times K_2}(\mathbf{c}_2^\kappa) \\ &= \sum_{\mu \in \widehat{K}} \mathcal{Q}_{K_1}^\Phi(M < R') \otimes (V_\mu^{K_1})^* \otimes V_\mu^{K_2} \end{aligned}$$

Taking the K_1 -invariant completes the proof of the second point. \square

Finally we have proved that the generalized character $[\text{Index}_{V_{R'}}^{K_1 \times K_2}(\mathbf{c}^{\kappa_\rho}|_{V_{R'}})]^{K_1}$ is equal to $\mathcal{Q}_{K_2}^\Phi(M < R')$. Taking the limit $R' \rightarrow \infty$ gives

$$\begin{aligned} \left[\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \text{T}^*K) \right]^{K_1} &= \lim_{R' \rightarrow \infty} \left[\text{Index}_{V_{R'}}^{K_1 \times K_2}(\mathbf{c}^{\kappa_\rho}|_{V_{R'}}) \right]^{K_1} \\ &= \lim_{R' \rightarrow \infty} \mathcal{Q}_{K_2}^\Phi(M < R') = \mathcal{Q}_{K_2}^\Phi(M). \end{aligned}$$

3.2. Proof of Proposition 3.3. We work here with the Hamiltonian action of $K_1 \times K_2$ on $M \times \mathcal{X}_P$. The action is $(k_1, k_2) \cdot (m, x) = (k \cdot m, k_2 \cdot x \cdot k_1^{-1})$ and the corresponding moment map is $\Phi = (\Phi_1, \Phi_2)$ with $\Phi_1(m, x) = \Phi_M(m) + \Phi_r(x)$ and $\Phi_2(m, x) = \Phi_l(x)$. Let $\|(\xi_1, \xi_2)\|_\rho^2 = \|\xi_1\|^2 + \rho\|\xi_2\|^2$ be the Euclidean norm $\mathfrak{k}_1^* \times \mathfrak{k}_2^*$ attached to $\rho > 0$.

Let us consider the quantization of $M \times \mathcal{X}_P$ via the map $\|\Phi\|_\rho^2$:

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) \in R^{-\infty}(K_1 \times K_2)$$

The critical set $\text{Cr}(\|\Phi\|_\rho^2)$ admits the decomposition

$$(3.41) \quad \text{Cr}(\|\Phi\|_\rho^2) = \bigcup_{\gamma \in \mathcal{B}_\rho} K_1 \times K_2 \cdot \mathcal{C}_\gamma$$

where $(m, x) \in \mathcal{C}_\gamma$ if and only if $\gamma = (\gamma_1, \gamma_2)$ with

$$(3.42) \quad \begin{cases} \Phi_M(m) + \Phi_r(x) = \gamma_1 \\ \Phi_l(x) = \gamma_2 \\ \tilde{\gamma}_1 \cdot m = 0 \\ \tilde{\gamma}_1 \cdot_r x + \rho \tilde{\gamma}_2 \cdot_l x = 0. \end{cases}$$

We have

$$(3.43) \quad \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) = \sum_{\gamma \in \mathcal{B}_\rho} \mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P)$$

where the generalized character $\mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P)$ is computed as an index of a transversally elliptic symbol in a neighborhood of

$$K_1 \times K_2 \cdot \mathcal{C}_\gamma \subset M \times \Phi_l^{-1}(K_2 \cdot \gamma_2).$$

Thanks to Theorem 2.9 we know that the support of the generalized character $\mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P)$ is contained in $\{(a, b) \in \widehat{K_1} \times \widehat{K_2} \mid \|a\|^2 + \rho\|b\|^2 \geq \|\gamma\|_\rho^2\}$. Hence

$$\text{support} \left([\mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P)]^{K_1} \right) \subset \left\{ b \in \widehat{K_2} \mid \rho\|b\|^2 \geq \|\gamma\|_\rho^2 \right\}$$

Let $r_P = \inf_{\xi \in \partial P} \|\xi\|$. We know then that

$$\left[\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) \right]^{K_1} = \sum_{\substack{\gamma \in \mathcal{B}_\rho \\ \|\gamma\|_\rho^2 < \rho r_P^2}} \left[\mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P) \right]^{K_1} + O(r_P).$$

Let $R_P < \rho r_P^2$ be a regular value of $\|\Phi\|_\rho^2 : M \times \mathcal{X}_P \rightarrow \mathbb{R}$ such that for all $\gamma \in \mathcal{B}_\rho$ we have $\|\gamma\|_\rho^2 < \rho r_P^2 \iff \|\gamma\|_\rho^2 < R_P$. Then

$$(3.44) \quad \left[\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) \right]^{K_1} = \left[\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}((M \times \mathcal{X}_P)_{< R_P}) \right]^{K_1} + O(r_P).$$

For the generalized index $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K)$ we have also a decomposition

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K) = \sum_{\gamma \in \mathcal{B}'_\rho} \mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times T^*K)$$

where \mathcal{B}'_ρ parametrizes the critical set of $\|\Phi\|_\rho^2 : M \times T^*K \rightarrow \mathbb{R}$. Like before we get

$$(3.45) \quad \left[\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K) \right]^{K_1} = \left[\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}((M \times T^*K)_{< R'_P}) \right]^{K_1} + O(r_P).$$

Here $R'_P < \rho r_P^2$ is a regular value of $\|\Phi\|_\rho^2 : M \times T^*K \rightarrow \mathbb{R}$ such that for all $\gamma \in \mathcal{B}'_\rho$ we have $\|\gamma\|_\rho^2 < \rho r_P^2 \iff \|\gamma\|_\rho^2 < R'_P$.

Lemma 3.6. *We have*

$$(3.46) \quad \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}((M \times \mathcal{X}_P)_{< R_P}) = \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}((M \times T^*K)_{< R'_P}).$$

Proof. The Lemma will follow from Proposition 2.6. We take here $V' = M \times \mathcal{X}_P^\rho$, $V = M \times K \times \mathcal{U}_P \subset M \times T^*K$ and the equivariant diffeomorphism $\Psi : V \rightarrow V'$ is equal to $\text{Id} \times \Upsilon$ where Υ was introduced in Section 2.5. Note that Ψ satisfies points (1) – (3) of Proposition 2.6.

Note that $\|\Phi(m, x)\|_\rho^2 < \rho r_P^2$ implies that $\|\Phi_l(x)\| < r_P$ and then $x \in \mathcal{X}_P^\rho$. Hence the open subset $U' := (M \times \mathcal{X}_P)_{< R_P}$ is contained in $V' = M \times \mathcal{X}_P^\rho$. In the same way the open subset $U := (M \times T^*K)_{< R'_P}$ is contained in V . We have $\Psi(U) = U'$ if $R_P = R'_P$.

We have proved that (3.46) is a consequence of Proposition 2.6. \square

Finally, if we take the difference between (3.44) and (3.45), we get

$$\left[\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) \right]^{K_1} - \left[\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K) \right]^{K_1} = O(r_P).$$

which is the relation of Proposition 3.3.

3.3. Proof of Proposition 3.4. Here we want to compare the K_1 -invariant part of the characters $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P)$ and $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M \times \mathcal{X}_P)$.

We know after Theorem 2.14 that

$$\begin{aligned} \mathcal{Q}_{K_2}(M_P) &= \left[\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M \times \mathcal{X}_P) \right]^{K_1} \\ &= \left[\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(U_\epsilon) \right]^{K_1} \end{aligned}$$

when $\epsilon > 0$ is any regular value of $\|\Phi_1\|^2$, and $U_\epsilon := \{\|\Phi_1\|^2 < \epsilon\} \subset M \times \mathcal{X}_P$.

In this section we fix once for all $\epsilon > 0$ small enough so that

$$(3.47) \quad \text{Cr}(\|\Phi_1\|^2) \cap \{\|\Phi_1\|^2 \leq \epsilon\} = \{\Phi_1 = 0\}.$$

Let \mathbf{c}_1 be the symbol $\text{Thom}(M, J_1) \otimes L$ attached to the prequantized Hamiltonian K_1 -manifold (M, Ω) . Let \mathbf{c}_3 be the symbol $\text{Thom}(\mathcal{X}_P, J_3) \otimes L_P$ attached to the prequantized Hamiltonian $K_1 \times K_2$ -manifold \mathcal{X}_P . The product $\mathbf{c} = \mathbf{c}_1 \odot \mathbf{c}_3$ corresponds to the symbol $\text{Thom}(N, J) \otimes L$ on $N = M \times \mathcal{X}_P$.

Let κ_0 and κ_ρ be the Kirwan vector fields associated to the functions $\|\Phi_1\|^2$ and $\|\Phi\|_\rho^2$ on $M \times \mathcal{X}_P$:

$$\kappa_0(m, x) = \left(\underbrace{\Phi_1(m, x) \cdot m}_{\kappa_I}; \underbrace{\Phi_1(m, x) \cdot_r x}_{\kappa_{II}} \right), \quad \kappa_\rho(m, x) = \kappa^0(m, x) + \rho(0, \underbrace{\Phi_l(x) \cdot_l x}_{\kappa_{III}}).$$

Let \mathbf{c}^{κ_ρ} be the symbol \mathbf{c} pushed by the vector field κ_ρ : we have

$$\mathbf{c}^{\kappa_\rho}(v; \eta) = \mathbf{c}_1(v - \kappa_I) \odot \mathbf{c}_3(\eta - \kappa_{II} - \rho \kappa_{III})$$

for $(v; \eta) \in T_{(m, x)}(M \times \mathcal{X}_P)$.

The character $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(U_\epsilon)$ is given by the index of the K_1 -transversally elliptic symbol $\mathbf{c}^{\kappa_0}|_{U_\epsilon}$. The character $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P)$ is given by the index of the $K_1 \times K_2$ -transversally elliptic symbol \mathbf{c}^{κ_ρ} .

Lemma 3.7. • *There exists $\rho(\epsilon) > 0$ such that*

$$\text{Cr}(\|\Phi\|_\rho^2) \cap \{\|\Phi_1\|^2 \leq \epsilon\} \subset \left\{ \|\Phi_1\|^2 \leq \frac{\epsilon}{2} \right\}$$

for any $0 \leq \rho \leq \rho(\epsilon)$.

Proof. With the help of Riemannian metrics on M and \mathcal{X}_P we define

$$\begin{aligned} a(\epsilon) &:= \inf_{\epsilon/2 \leq \|\Phi_1(m, x)\| \leq \epsilon} \|\kappa^0(m, x)\| \\ b &:= \sup_{x \in \mathcal{X}_P} \|\Phi_l(x) \cdot_l x\|. \end{aligned}$$

We have $a(\epsilon) > 0$ thanks to (3.47), and $b < \infty$ since \mathcal{X}_P is compact. It is now easy to check that $\{\kappa_\rho = 0\} \cap \{\epsilon/2 \leq \|\Phi_1\|^2 \leq \epsilon\} = \emptyset$ if $0 \leq \rho < \frac{a(\epsilon)}{b}$. \square

The symbols $\mathbf{c}^{\kappa_\rho}|_{U_\epsilon}$, $\rho \in [0, \rho(\epsilon)]$ are $K_1 \times K_2$ -transversally elliptic, and they define the same class in $\mathbf{K}_{K_1 \times K_2}(T_{K_1 \times K_2} U_\epsilon)$. Hence $\mathcal{Q}_{K_2}(M_P)$ can be computed as the K_1 -invariant part of

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(U_\epsilon) := \text{Index}_{K_1 \times K_2}^{U_\epsilon}(\mathbf{c}^{\kappa_\rho}|_{U_\epsilon}) \in R^{-\infty}(K_1 \times K_2)$$

for $\rho \in [0, \rho(\epsilon)]$.

A component $K_1 \times K_2 \cdot \mathcal{C}_\gamma$ of $\text{Cr}(\|\Phi\|_\rho^2)$ is contained in U_ϵ if and only $\|\gamma_1\| < \epsilon$: hence the decomposition (3.43) for the character $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P)$ gives

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) = \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(\mathcal{U}_\epsilon) + \sum_{\substack{\gamma \in \mathcal{B}_\rho \\ \|\gamma_1\|^2 \geq \epsilon}} \mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P).$$

where

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(\mathcal{U}_\epsilon) = \sum_{\substack{\gamma \in \mathcal{B}_\rho \\ \|\gamma_1\|^2 < \epsilon}} \mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P).$$

Taking the K_1 -invariant gives

$$(3.48) \quad [\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P)]^{K_1} = \mathcal{Q}_{K_2}(M_P) + \sum_{\substack{\gamma \in \mathcal{B}_\rho \\ \|\gamma_1\|^2 \geq \epsilon}} [\mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P)]^{K_1}.$$

In general we know that the support of the generalized character $[\mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P)]^{K_1}$ is included in $\{b \in \widehat{K_2} \mid \rho\|b\|^2 \geq \|\gamma_1\|^2 + \rho\|\gamma_2\|^2\}$. When $\|\gamma_1\|^2 \geq \epsilon$ we have then that the support of $[\mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P)]^{K_1}$ is contained in $\{b \in \widehat{K_2} \mid \rho\|b\|^2 \geq \epsilon\}$.

Finally (3.48) imposes that

$$[\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P)]^{K_1} = \mathcal{Q}_{K_2}(M_P) + O((\epsilon/\rho)^{1/2}).$$

when $0 < \rho \leq \rho(\epsilon)$, which is the precise content of Proposition 3.4.

4. OTHER PROPERTIES OF \mathcal{Q}^Φ

Let (M, ω, Φ) be a proper Hamiltonian K -manifold which is prequantized by a line bundle L . The character $\mathcal{Q}_K^\Phi(M)$ is computed by means of a scalar product on \mathfrak{k}^* . The fact that $\mathcal{Q}_K^\Phi(M) = \mathcal{Q}_K^{-\infty}(M)$ gives the following

Proposition 4.1. *The character $\mathcal{Q}_K^\Phi(M)$ does not depend of the choice of a scalar product on \mathfrak{k}^**

In this section we work in the setting where $K = K_1 \times K_2$. Let Φ_1 be the moment map relative to the K_1 -action.

4.1. Φ_1 is proper. In this subsection we suppose that the moment map Φ_1 relative to the K_1 -action is *proper*. We fix an invariant Euclidean norm $\|\bullet\|^2$ on \mathfrak{k} in such a way that $\mathfrak{k}_1 = \mathfrak{k}_2^\perp$.

Let us “quantize” (M, Ω) via the invariant proper function $\|\Phi_1\|^2$: let

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M) \in R^{-\infty}(K_1 \times K_2)$$

be the corresponding generalized character.

Theorem 4.2. *We have*

$$(4.49) \quad \mathcal{Q}_{K_1 \times K_2}^\Phi(M) = \mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M) \quad \text{in} \quad R^{-\infty}(K_1 \times K_2).$$

Proof. On $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ we may consider the family of invariant Euclidean norms: $\|X_1 \oplus X_2\|_\rho^2 = \|X_1\|^2 + \rho\|X_2\|^2$ for $X_j \in \mathfrak{k}_j$. Let

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M) \in R^{-\infty}(K_1 \times K_2)$$

be the quantization of M computed via the map $\|\Phi\|_\rho^2 = \|\Phi_1\|^2 + \rho\|\Phi_2\|^2$. By definition, $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M)$ is equal to $\mathcal{Q}_{K_1 \times K_2}^{\Phi, 0}(M)$, and we know after Proposition 4.1 that $\mathcal{Q}_{K_1 \times K_2}^{\Phi}(M)$ coincides with the generalized character $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M) \in R^{-\infty}(K)$ for any $\rho > 0$.

Let us prove that $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M) = \mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M)$. We denote $O(r) \in R^{-\infty}(K_1 \times K_2)$ any generalized character supported outside the ball

$$\{\xi \in \mathfrak{k}_1^* \times \mathfrak{k}_2^* \mid \|\xi_1\|^2 + \|\xi_2\|^2 < r^2\}.$$

And we denote $O_1(r) \in R^{-\infty}(K_1 \times K_2)$ any generalized character supported outside the

$$\{\xi \in \mathfrak{k}_1^* \times \mathfrak{k}_2^* \mid \|\xi_1\| < r\}.$$

Let $R_1 > 0$ be a regular value of $\|\Phi_1\|^2$: the open subset $\{\|\Phi_1\|^2 < R_1\}$ is denoted $M_{<R_1}$. We know that

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M) = \mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M_{<R_1}) + O_1(\sqrt{R_1}).$$

Like in the Lemma 3.7, we know that

$$(4.50) \quad \text{Cr}(\|\Phi\|_\rho^2) \cap \{\|\Phi_1\|^2 = R_1\} = \emptyset.$$

for $\rho \geq 0$ small enough. The identity (4.50) first implies that

$$\begin{aligned} \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M) &= \sum_{\substack{\gamma \in \mathcal{B}_\rho \\ \|\gamma_1\|^2 < R_1}} \mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M) + \sum_{\substack{\gamma \in \mathcal{B}_\rho \\ \|\gamma_1\|^2 > R_1}} \mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M) \\ &= \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M_{<R_1}) + O(\sqrt{R_1}). \end{aligned}$$

In the second equality we have used that $\mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M) = O(\sqrt{R_1})$ when $\|\gamma_1\|^2 > R_1$ since the ball $\{(\xi_1, \xi_2) \in \mathfrak{k}_1^* \times \mathfrak{k}_2^* \mid \|\xi_1\|^2 + \|\xi_2\|^2 < R_1\}$ is contained in

$$\{(\xi_1, \xi_2) \in \mathfrak{k}_1^* \times \mathfrak{k}_2^* \mid \|(\xi_1, \xi_2)\|_\rho^2 < \|(\gamma_1, \gamma_2)\|_\rho^2\}.$$

The identity (4.50) shows also that the symbol $\mathbf{c}^{\kappa_\rho}|_{M_{<R_1}}$ are homotopic for $\rho \geq 0$ small enough. Hence

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M_{<R_1}) = \mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M_{<R_1})$$

We get finally that $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M) - \mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M) = O(\sqrt{R_1}) + O_1(\sqrt{R_1})$ for any regular value R_1 of $\|\Phi_1\|^2$. We have proved that $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M) - \mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M) = 0$. \square

Let us explain how Theorem 4.2 contains the identity that we called “*quantization commutes with reduction in the singular setting*” in [21]. By definition the K_1 -invariant part of the right hand side of (4.49) is equal to the geometric quantization of the (possibly singular) compact Hamiltonian K_2 -manifold

$$M//_0 K_1 := \Phi_1^{-1}(0)/K_1.$$

Using now the fact that the left hand side of (4.49) is equal to $\mathcal{Q}_{K_1 \times K_2}^{-\infty}(M)$, we see that the multiplicity of $V_\mu^{K_2}$ in $\mathcal{Q}_{K_2}(M//_0 K_1)$ is equal to the geometric quantization of the (possibly singular) compact manifold

$$M \times \overline{K_2 \cdot \mu} //_{(0, \mu)} K_1 \times K_2.$$

4.2. The symplectic reduction $M//_0 K_1$ is smooth. Let (M, Ω) be an Hamiltonian $K_1 \times K_2$ -manifold with a proper moment map $\Phi = (\Phi_1, \Phi_2)$. In this section we suppose that 0 is a regular value of Φ_1 and that K_1 acts freely on $\Phi_1^{-1}(0)$. We work then with the (smooth) Hamiltonian K_2 -manifold

$$N := \Phi_1^{-1}(0)/K_1.$$

We still denote by $\Phi_2 : N \rightarrow \mathfrak{k}_2^*$ the moment map relative to the K_2 -action: note that this map is proper. Hence we can quantize the K_2 -action on N via the map Φ_2 . Let $\mathcal{Q}_{K_2}^{\Phi_2}(N) \in R^{-\infty}(K_2)$ be the corresponding character.

Proposition 4.3. *We have*

$$(4.51) \quad [\mathcal{Q}_{K_1 \times K_2}^{\Phi}(M)]^{K_1} = \mathcal{Q}_{K_2}^{\Phi_2}(N) \quad \text{in } R^{-\infty}(K_2).$$

Proof. When Φ_1 is proper, the manifold N is compact. Then the right hand side of (4.51) is equal to $\mathcal{Q}_{K_2}(N)$, and we know from Theorem 4.2 that the left hand side of (4.51) is equal to $[\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M)]^{K_1}$. In this case (4.51) becomes $[\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M)]^{K_1} = \mathcal{Q}_{K_2}(M//_0 K_1)$ which is the content of Theorem 2.13.

Let us consider the general case where Φ_1 is not proper. Thanks to Theorem 1.4 one knows that the multiplicities of $V_\mu^{K_2}$ in $[\mathcal{Q}_{K_1 \times K_2}^{\Phi}(M)]^{K_1}$ and $\mathcal{Q}_{K_2}^{\Phi_2}(N)$ are respectively equal to the quantization of the (possibly singular) symplectic reductions

$$\mathcal{M}_\mu := M \times \overline{K_2 \cdot \mu} //_{(0,0)} K_1 \times K_2.$$

and

$$\mathcal{M}'_\mu := N \times \overline{K_2 \cdot \mu} //_0 K_2, \quad \text{with } N = M//_0 K_1.$$

Note that \mathcal{M}_μ and \mathcal{M}'_μ coincide as symplectic reduced space. Let us prove that their geometric quantization are identical also. The proof will be done for $\mu = 0$: the other case follows from the shifting trick.

Let \mathbf{c} be the $K_1 \times K_2$ -equivariant symbol $\text{Thom}(M, J) \otimes L_M$. Let κ be the Kirwan vector field attached to the moment map $\Phi = (\Phi_1, \Phi_2)$. Let \mathbf{c}^κ be the symbol \mathbf{c} pushed by κ . Let us denote $M_{<\epsilon}$ the open subset $\{\|\Phi\|^2 < \epsilon\}$. For $\epsilon > 0$ small enough, the symbol $\mathbf{c}^\kappa|_{M_{<\epsilon}}$ is $K_1 \times K_2$ -transversally elliptic, and $\mathcal{Q}(\mathcal{M}_0)$ is the $K_1 \times K_2$ -invariant part of $\text{Index}_{M_{<\epsilon}}^{K_1 \times K_2}(\mathbf{c}^\kappa|_{M_{<\epsilon}})$.

Let \mathbf{c}_2 be the K_2 -equivariant symbol $\text{Thom}(N, J) \otimes L_N$. Let κ_2 be the Kirwan vector field attached to the moment map Φ_2 . Let $\mathbf{c}_2^{\kappa_2}$ be the symbol \mathbf{c}_2 pushed by κ_2 . Let us denote $N_{<\epsilon}$ the open subset $\{\|\Phi_2\|^2 < \epsilon\}$. For $\epsilon > 0$ small enough, the symbol $\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}}$ is K_2 -transversally elliptic, and $\mathcal{Q}(\mathcal{M}'_0)$ is the K_2 -invariant part of $\text{Index}_{N_{<\epsilon}}^{K_2}(\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}})$.

Our proof follows from the comparison of the classes

$$[\mathbf{c}^\kappa|_{M_{<\epsilon}}] \in \mathbf{K}_{K_1 \times K_2}(\text{T}_{K_1 \times K_2} M_{<\epsilon})$$

and

$$[\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}}] \in \mathbf{K}_{K_2}(\text{T}_{K_2} N_{<\epsilon})$$

A neighborhood of the smooth submanifold $Z := \Phi_1^{-1}(0)$ in M is diffeomorphic to a neighborhood of the 0-section of the bundle $Z \times \mathfrak{k}_1^* \rightarrow Z$. Let $Z_{<\epsilon} = Z \cap M_{<\epsilon}$ so that $N_{<\epsilon} = Z_{<\epsilon}/K_1$. Hence $[\mathbf{c}^\kappa|_{M_{<\epsilon}}]$ can be seen naturally a class in the K-group $\mathbf{K}_{K_1 \times K_2}(\mathbf{T}_{K_1 \times K_2}(Z_{<\epsilon} \times \mathfrak{k}_1^*))$.

Following Atiyah [1][Theorem 4.3], the inclusion map $j : Z_{<\epsilon} \hookrightarrow Z_{<\epsilon} \times \mathfrak{k}_1^*$ induces the Thom isomorphism

$$j! : \mathbf{K}_{K_1 \times K_2}(\mathbf{T}_{K_1 \times K_2} Z_{<\epsilon}) \longrightarrow \mathbf{K}_{K_1 \times K_2}(\mathbf{T}_{K_1 \times K_2}(Z_{<\epsilon} \times \mathfrak{k}_1^*)),$$

with the commutative diagram

$$(4.52) \quad \begin{array}{ccc} \mathbf{K}_{K_1 \times K_2}(\mathbf{T}_{K_1 \times K_2} Z_{<\epsilon}) & \xrightarrow{j!} & \mathbf{K}_{K_1 \times K_2}(\mathbf{T}_{K_1 \times K_2}(Z_{<\epsilon} \times \mathfrak{k}_1^*)) \\ & \searrow \text{Index}_{Z_{<\epsilon}}^{K_1 \times K_2} & \downarrow \text{Index}_{Z_{<\epsilon} \times \mathfrak{k}_1^*}^{K_1 \times K_2} \\ & & R^{-\infty}(K_1 \times K_2) \end{array}$$

Let $\pi_1 : Z_{<\epsilon} \rightarrow N_{<\epsilon}$ be the quotient relative to the free action of K_1 . The corresponding isomorphism

$$\pi_1^* : \mathbf{K}_{K_2}(\mathbf{T}_{K_2} N_{<\epsilon}) \longrightarrow \mathbf{K}_{K_1 \times K_2}(\mathbf{T}_{K_1 \times K_2} Z_{<\epsilon})$$

satisfies the following rule :

$$(4.53) \quad \left[\text{Index}_{K_1 \times K_2}^{Z_{<\epsilon}}(\pi_1^* \theta) \right]^{K_1} = \text{Index}_{K_2}^{N_{<\epsilon}}(\theta)$$

for any $\theta \in \mathbf{K}_{K_2}(\mathbf{T}_{K_2} N_{<\epsilon})$.

Lemma 4.4 ([19]). *We have*

$$j! \circ \pi_1^* \left([\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}}] \right) = [\mathbf{c}^\kappa|_{M_{<\epsilon}}]$$

in $\mathbf{K}_{K_1 \times K_2}(\mathbf{T}_{K_1 \times K_2}(Z_{<\epsilon} \times \mathfrak{k}_1^*))$.

Proof. This Lemma is proven in [19][Section 6.2] when the group K_2 is trivial. It is easy to check that the proof extends naturally to our setting. \square

If one uses Lemma 4.4 together with (4.52) and (4.53), we get that

$$\begin{aligned} \mathcal{Q}(\mathcal{M}_0) &= \left[\text{Index}_{Z_{<\epsilon} \times \mathfrak{k}_1^*}^{K_1 \times K_2}(\mathbf{c}^\kappa|_{M_{<\epsilon}}) \right]^{K_1 \times K_2} \\ &= \left[\text{Index}_{N_{<\epsilon}}^{K_2}(\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}}) \right]^{K_2} = \mathcal{Q}(\mathcal{M}'_0). \end{aligned}$$

\square

5. EXAMPLE: THE COTANGENT BUNDLE OF AN ORBIT

5.1. The formal quantization of \mathbf{T}^*K . Let K be a compact connected Lie group equipped with the action of two copies of K : $(k_1, k_2) \cdot a = k_2 a k_1^{-1}$. Then we have a Hamiltonian action of $K_1 \times K_2$ on the cotangent bundle \mathbf{T}^*K . In this section, we check that each formal geometric quantization of \mathbf{T}^*K , $\mathcal{Q}_{K_1 \times K_2}^{-\infty}(\mathbf{T}^*K)$ and $\mathcal{Q}_{K_1 \times K_2}^\Phi(\mathbf{T}^*K)$, are both equal to the $K_1 \times K_2$ -module $L^2(K)$.

The tangent bundle TK is identified with $K \times \mathfrak{k}$ through the right translations: to $(a, X) \in K \times \mathfrak{k}$ we associate $\frac{d}{dt}ae^{tX}|_0$. The action of $K_1 \times K_2$ on the cotangent bundle $T^*K \simeq K \times \mathfrak{k}^*$ is then

$$(k_1, k_2) \cdot (a, \xi) = (k_2 a k_1^{-1}, k_1 \cdot \xi).$$

The symplectic form on T^*K is $\Omega := -d\lambda$, where λ is the Liouville 1-form. Let us compute these two form in coordinates. The tangent bundle of $T^*K \simeq K \times \mathfrak{k}^*$ is identified with $T^*K \times \mathfrak{k} \times \mathfrak{k}^*$: for each $(a, \xi) \in T^*K$, we have a two form $\Omega_{(a, \xi)}$ on $\mathfrak{k} \times \mathfrak{k}^*$. A direct computation gives

$$\Omega_{(a, \xi)}(X, Y) = \langle \xi, [X, Y] \rangle, \quad \Omega_{(a, \xi)}(\eta, \eta') = 0, \quad \Omega_{(a, \xi)}(X, \eta) = \langle \eta, X \rangle$$

for $X, Y \in \mathfrak{k}$ and $\eta, \eta' \in \mathfrak{k}^*$. So $\Omega_{(a, \xi)} = \Omega_0 + \pi_\xi$ where Ω_0 is the canonical (constant) symplectic form on $\mathfrak{k} \times \mathfrak{k}^*$ and π_ξ is the closed two form on \mathfrak{k} defined by $\pi_\xi(X, Y) = \langle \xi, [X, Y] \rangle$.

If we identify $\mathfrak{k} \simeq \mathfrak{k}^*$ through an invariant Euclidean norm, the symplectic structure on $T_{(a, \xi)}(T^*K) \simeq \mathfrak{k} \times \mathfrak{k}^*$ is given by a skew-symmetric matrix

$$A_\xi := \begin{pmatrix} \text{ad}(\xi) & I_n \\ -I_n & 0 \end{pmatrix}.$$

We will work with the following compatible almost complex structure on the tangent bundle of T^*K : $J_\xi = -A_\xi(-A_\xi^2)^{-1/2}$. When $\xi = 0$, the complex structure J_0 on $\mathfrak{k} \times \mathfrak{k}^*$ is defined by the matrix

$$J_0 := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Hence the complex K -module $(\mathfrak{k} \times \mathfrak{k}^*, J_0)$ is naturally identified with the complexification $\mathfrak{k}_\mathbb{C}$ of \mathfrak{k} .

One checks easily that the moment map relative to the $K_1 \times K_2$ -action is the *proper* map $\Phi : T^*K \rightarrow \mathfrak{k}_1^* \times \mathfrak{k}_2^*$ defined by $\Phi(a, \xi) = (-\xi, a \cdot \xi)$.

Here the symplectic manifold T^*K is prequantized by the trivial line bundle.

5.1.1. *Computation of $\mathcal{Q}_{K_1 \times K_2}^{-\infty}(T^*K)$.* Let $\mathcal{O}_1 \times \mathcal{O}_2$ be a coadjoint orbit of $K_1 \times K_2$ in $\mathfrak{k}_1^* \times \mathfrak{k}_2^*$. One checks that

$$(5.54) \quad \Phi^{-1}(\mathcal{O}_1 \times \mathcal{O}_2) = \begin{cases} \emptyset & \text{if } \mathcal{O}_1 \neq -\mathcal{O}_2 \\ \text{a } K_1 \times K_2 \text{-orbit} & \text{if } \mathcal{O}_1 = -\mathcal{O}_2. \end{cases}$$

We know that the stabiliser subgroup K_ξ of an element $\xi \in \mathfrak{k}^*$ is connected. Then the stabilizer subgroup $(K_1 \times K_2)_{(a, \xi)} = \{(k_1, a k_1 a^{-1}), k_1 \in K_\xi\}$ is also connected.

Let $(T^*K)_{(\mu, \lambda)}$ be the symplectic reduction of T^*K at the level $(\mu, \lambda) \in \widehat{K}^2$. For any $\mu \in \widehat{K}$, we define $\mu^* \in \widehat{K}$ by the relation $-K \cdot \mu = K \cdot \mu^*$: note that $V_{\mu^*}^K \simeq (V_\mu^K)^*$. If one uses Theorem 2.16, one has

$$(5.55) \quad \mathcal{Q}((T^*K)_{(\mu, \lambda)}) = \begin{cases} 0 & \text{if } \lambda \neq \mu^* \\ 1 & \text{if } \lambda = \mu^*. \end{cases}$$

Finally

$$\begin{aligned}\mathcal{Q}_{K_1 \times K_2}^{-\infty}(\mathrm{T}^*K) &= \sum_{(\mu, \lambda) \in \widehat{K} \times \widehat{K}} \mathcal{Q}((\mathrm{T}^*K)_{(\mu, \lambda)}) V_\mu^{K_1} \otimes V_\lambda^{K_2} \\ &= \sum_{\mu \in \widehat{K}} V_\mu^{K_1} \otimes (V_\mu^{K_2})^* = L^2(K).\end{aligned}$$

5.1.2. *Computation of $\mathcal{Q}_{K_1 \times K_2}^\Phi(\mathrm{T}^*K)$.* The Kirwan vector field on T^*K is

$$\kappa(a, \xi) = -2\xi \in \mathfrak{k}_\mathbb{C}.$$

Let \mathbf{c}^κ be the symbol $\mathrm{Thom}(\mathrm{T}^*K, J)$ pushed by the vector field $\frac{1}{2}\kappa$. At each $(a, \xi) \in \mathrm{T}^*K$, the map $\mathbf{c}_{(a, \xi)}^\kappa(X \oplus \eta)$ from $\wedge_{J_\xi}^{\mathrm{even}}(\mathfrak{k} \times \mathfrak{k}^*)$ to $\wedge_{J_\xi}^{\mathrm{odd}}(\mathfrak{k} \times \mathfrak{k}^*)$ is equal to the clifford map $\mathbf{c}(X + \xi \oplus \eta)$. Note that \mathbf{c}^κ is a K_2 -transversally elliptic symbol on T^*K : we have $\mathrm{Char}(\mathbf{c}^\kappa) \cap \mathrm{T}_{K_2}(\mathrm{T}^*K) = \{(1, 0)\}$. We will now compute the equivariant index of \mathbf{c}^κ .

First we consider the homotopy $t \in [0, 1] \rightarrow J_{t\xi}$ of symplectic structure on T^*K . Let $\tilde{\mathbf{c}}^\kappa$ be the symbol acting on $\wedge_{J_0}^\bullet(\mathfrak{k} \times \mathfrak{k}^*) = \wedge_\mathbb{C}^\bullet \mathfrak{k}_\mathbb{C}$. Proposition 2.6 shows that the symbols \mathbf{c}^κ and $\tilde{\mathbf{c}}^\kappa$ define the same class in $K_{K_1 \times K_2}(\mathrm{T}_{K_2}(\mathrm{T}^*K))$.

The projection $\pi : \mathrm{T}^*K \rightarrow \mathfrak{k}^*$ corresponds to the quotient map relative to the free action of K_2 . At the level of K -groups we get an isomorphism

$$\pi_* : K_{K_1 \times K_2}(\mathrm{T}_{K_2}(\mathrm{T}^*K)) \rightarrow K_{K_1}(\mathrm{T}\mathfrak{k}^*).$$

Atiyah [1] proves that

$$\mathrm{Index}_{K_1 \times K_2}^{\mathrm{T}^*K}(\sigma) = \sum_{\mu \in \widehat{K}} \mathrm{Index}_{K_1}^{\mathfrak{k}^*}(\pi_*(\sigma \otimes V_\mu^{K_2})) \otimes (V_\mu^{K_2})^*$$

for any class $\sigma \in K_{K_1 \times K_2}(\mathrm{T}_{K_2}(\mathrm{T}^*K))$. In our case the symbol $\pi^*(\tilde{\mathbf{c}}^\kappa)$ is equal to the Bott symbol $\mathrm{Bott}(\mathfrak{k}^*)$, and for any K_2 -module E_2 we have

$$\pi_*(\tilde{\mathbf{c}}^\kappa \otimes E_2) = \mathrm{Bott}(\mathfrak{k}^*) \otimes E_1$$

where E_1 is the module E_2 with the action of K_1 . Then

$$\begin{aligned}\mathcal{Q}_{K_1 \times K_2}^\Phi(\mathrm{T}^*K) &= \mathrm{Index}_{K_1 \times K_2}^{\mathrm{T}^*K}(\tilde{\mathbf{c}}^\kappa) \\ &= \sum_{\mu \in \widehat{K}} \mathrm{Index}_{K_1}^{\mathfrak{k}^*}(\mathrm{Bott}(\mathfrak{k}^*) \otimes V_\mu^{K_1}) \otimes (V_\mu^{K_2})^* \\ &= \sum_{\mu \in \widehat{K}} V_\mu^{K_1} \otimes (V_\mu^{K_2})^* = L^2(K),\end{aligned}$$

since $\mathrm{Index}_{K_1}^{\mathfrak{k}^*}(\mathrm{Bott}(\mathfrak{k}^*)) = 1$.

5.2. **The formal quantization of $\mathrm{T}^*(K/H)$.** Let H be a closed connected subgroup of K . Look at T^*K as a Hamiltonian manifold relatively to the action of $H \times K \subset K_1 \times K_2$. The moment map $\Phi = (\Phi_H, \Phi_K)$ is defined by : $\Phi_H(a, \xi) = -\mathrm{pr}(\xi)$ and $\Phi_K(a, \xi) = a \cdot \xi$, where $\mathrm{pr} : \mathfrak{k}^* \rightarrow \mathfrak{h}^*$ is the projection. Note that Φ is a proper map.

The cotangent bundle $\mathrm{T}^*(K/H)$, viewed as K -manifold, is equal to the symplectic reduction of T^*K relatively to the H -action: if the kernel of the projection pr is denoted \mathfrak{h}^\perp , we have

$$\Phi_H^{-1}(0)/H = K \times_H \mathfrak{h}^\perp = \mathrm{T}^*(K/H).$$

We are here in the setting of section 4.2. The reduction of the $H \times K$ proper Hamiltonian manifold T^*K relatively to the H -action is smooth, then its formal quantization is computed as follows

$$\begin{aligned}
 \mathcal{Q}_K^\Phi(T^*(K/H)) &= [\mathcal{Q}_{H \times K}^\Phi(T^*K)]^H = [\mathcal{Q}_{K_1 \times K_2}^\Phi(T^*K)|_{H \times K}]^H \\
 (5.56) \qquad \qquad \qquad &= [L^2(K)]^H \\
 &= L^2(K/H).
 \end{aligned}$$

Here the fact that $\mathcal{Q}_{H \times K}^\Phi(T^*K)$ is equal to the restriction of $\mathcal{Q}_{K_1 \times K_2}^\Phi(T^*K) = L^2(K)$ to $H \times K$ is a consequence of Theorem 1.3.

Let us denote $\left[T^*(K/H)\right]_\mu$ the symplectic reduction at $\mu \in \widehat{K}$ of the K-Hamiltonian manifold $T^*(K/H)$. Theorem 1.4 together with (1.6) gives

Corollary 5.1. *For any $\mu \in \widehat{K}$, we have*

$$\mathcal{Q}\left(\left[T^*(K/H)\right]_\mu\right) = \dim [V_\mu^K]^H,$$

where $[V_\mu^K]^H$ is the subspace of H -invariant vector.

5.3. The formal quantization of $T^*(K/H)$ relatively to the action of G .

Let G be a closed connected subgroup of K . We look at the hamiltonian action of G on $T^*(K/H)$. Let $\Phi_G : T^*(K/H) \rightarrow \mathfrak{g}^*$ be the moment map. We consider also the restriction of the K-module $L^2(K/H)$ to G .

We have

Proposition 5.2. *The following statements are equivalent*

- (1) *The moment map $\Phi_G : T^*(K/H) \rightarrow \mathfrak{g}^*$ is proper.*
- (2) *$\Phi_G^{-1}(0)$ is equal to the zero section.*
- (3) *$k \cdot \mathfrak{g} + \mathfrak{h} = \mathfrak{k}$, for any $k \in K$.*
- (4) *$\mathfrak{g} + \mathfrak{h} = \mathfrak{k}$*
- (5) *G acts transitively on K/H .*
- (6) *$L^2(K/H)]^G \simeq \mathbb{C}$*
- (7) *$L^2(K/H)|_G$ is an admissible G -representation.*

Proof. (1) \implies (7) is a consequence of Theorem 1.3. Let us prove that (7) \implies (6). Suppose now that

$$L^2(K/H)|_G = \sum_{\mu \in \widehat{K}} [V_\mu^K]^H \otimes (V_\mu^K)^*|_G$$

is an admissible G -representation. It means that for any $\lambda \in \widehat{G}$ the set

$$A_\lambda := \left\{ \mu \in \widehat{K} \mid [V_\mu^K]^H \neq \{0\} \text{ and } [(V_\lambda^G)^* \otimes (V_\mu^K)^*|_G]^G \neq \{0\} \right\}$$

is finite. Then the vector space $L^2(K/H)]^G$ is equal to the finite dimensional vector space $\sum_{\mu \in A_0} [V_\mu^K]^H \otimes [(V_\mu^K)^*]^G$. It is not difficult to check that if $\mu \in A_0$, then $k\mu \in A_0$ for $k \gg 1$. Finally the fact that A_0 is finite implies that A_0 is reduced to $\mu = 0$. Hence the only G -invariant functions on K/H are the scalars.

(6) \iff (5) \iff (4) \iff (3) is a general fact concerning smooth actions of a compact connected Lie group G on a compact connected manifold M . The manifold M does not have G -invariant functions which are not scalar if and only if the action of G on M is transitive. And given a point $m \in M$, the orbit $G \cdot m$ is all of M if

and only if tangent spaces $T_m(G \cdot m)$ and $T_m M$ are equal. If we take $m = \overline{k^{-1}}$ in $M = K/H$, the condition $T_m(G \cdot m) = T_m M$ is equivalent to $k \cdot \mathfrak{g} + \mathfrak{h} = \mathfrak{k}$.

Let us check (3) \implies (2). Let $[k, \xi] \in K \times_H \mathfrak{h}^\perp = T^*(K/H)$. We have $\Phi_G([k, \xi]) = 0$ if and only if $k \cdot \xi \in \mathfrak{g}^\perp$. Hence the vector ξ belongs to

$$k^{-1} \cdot \mathfrak{g}^\perp \bigcap \mathfrak{h}^\perp = (k^{-1} \cdot \mathfrak{g} + \mathfrak{h})^\perp.$$

Hence condition (3) imposes that $\xi = 0$.

(2) \iff (1) comes from the fact that Φ_G is a homogeneous map of degree one between the vector bundle $T^*(K/H)$ and the vector space \mathfrak{g}^* . \square

Suppose now that the cotangent bundle $T^*(K/H)$ is a *proper* Hamiltonian G -manifold. Let us denote $[T^*(K/H)]_{\mu, G}$ the (compact) symplectic reduction at $\mu \in \widehat{G}$ of the G -Hamiltonian manifold $T^*(K/H)$. Then,

Corollary 5.3. *The multiplicity of V_μ^G in $L^2(K/H)$ is equal to the quantization of the reduced space $[T^*(K/H)]_{\mu, G}$.*

Proof. Using Theorem 1.3, equality (5.56) gives then

$$\begin{aligned} \mathcal{Q}_G^{-\infty}(T^*(K/H)) &= \mathcal{Q}_K^{-\infty}(T^*(K/H))|_G \\ &= L^2(K/H)|_G. \end{aligned}$$

In other words, the multiplicity of V_μ^G in $L^2(K/H)$ is equal to the quantization of the reduced space $[T^*(K/H)]_{\mu, G}$. \square

REFERENCES

- [1] M.F. ATIYAH, Elliptic operators and compact groups, Springer, 1974. Lecture notes in Mathematics, **401**.
- [2] M.F. ATIYAH, G.B. SEGAL, *The index of elliptic operators II*, Ann. Math. **87**, 1968, p. 531-545.
- [3] M.F. ATIYAH, I.M. SINGER, *The index of elliptic operators I*, Ann. Math. **87**, 1968, p. 484-530.
- [4] M.F. ATIYAH, I.M. SINGER, *The index of elliptic operators III*, Ann. Math. **87**, 1968, p. 546-604.
- [5] M.F. ATIYAH, I.M. SINGER, *The index of elliptic operators IV*, Ann. Math. **93**, 1971, p. 139-141.
- [6] N. BERLINE and M. VERGNE, The Chern character of a transversally elliptic symbol and the equivariant index, *Invent. Math.*, **124**, 1996, p. 11-49.
- [7] N. BERLINE and M. VERGNE, L'indice équivariant des opérateurs transversalement elliptiques, *Invent. Math.*, **124**, 1996, p. 51-101.
- [8] J. J. DUISTERMAAT, *The heat equation and the Lefschetz fixed point formula for the Spin^c-Dirac operator*, Progress in Nonlinear Differential Equation and Their Applications, vol. 18, Birkhauser, Boston, 1996.
- [9] V. GUILLEMIN and S. STERNBERG, Geometric quantization and multiplicities of group representations, *Invent. Math.*, **67**, 1982, p. 515-538.
- [10] V. GUILLEMIN and S. STERNBERG, A normal form for the moment map, in *Differential Geometric Methods in Mathematical Physics* (S. Sternberg, ed.), Reidel Publishing Company, Dordrecht, 1984.
- [11] V. GUILLEMIN and S. STERNBERG, Symplectic techniques in physics, Cambridge University Press, Cambridge, 1990.
- [12] B. KOSTANT, Quantization and unitary representations, in *Modern Analysis and Applications*, Lecture Notes in Math., Vol. 170, Springer-Verlag, 1970, p. 87-207.
- [13] E. LERMAN, Symplectic cut, *Math. Res. Lett.* **2**, 1995, p. 247-258.
- [14] E. LERMAN, E. MEINRENKEN, S. TOLMAN and C. WOODWARD, Non-Abelian convexity by symplectic cuts, *Topology*, **37**, 1998, p. 245-259.
- [15] X. MA, W. ZHANG, Geometric quantization for proper moment map, arXiv:0812.3989

- [16] E. MEINRENKEN, On Riemann-Roch formulas for multiplicities, *J. Amer. Math. Soc.*, **9**, 1996, p. 373-389.
- [17] E. MEINRENKEN, Symplectic surgery and the Spin^c -Dirac operator, *Advances in Math.*, **134**, 1998, p. 240-277.
- [18] E. MEINRENKEN, R. SJAMAAR, Singular reduction and quantization, *Topology*, **38**, 1999, p. 699-762.
- [19] P-E. PARADAN, Localization of the Riemann-Roch character, *J. Funct. Anal.* **187**, 2001, p. 442-509.
- [20] P-E. PARADAN, Spin^c quantization and the K -multiplicities of the discrete series, *Annales Scientifiques de l'E. N. S.*, **36**, 2003, p. 805-845.
- [21] P-E. PARADAN, Formal geometric quantization, *Ann. Inst. Fourier* **59**, 2009, p. 199-238.
- [22] P-E. PARADAN M. VERGNE, Index of transversally elliptic operators, 40 pages, to appear in *Astérisque, Soc. Math. Fr.*. Arxiv math/08041225.
- [23] P-E. PARADAN, Multiplicities of the discrete series, 38 pages. arXiv:0812.0059.
- [24] R. SJAMAAR, E. LERMAN, Stratified symplectic spaces and reduction, *Annals of Math.*, **134**, 1991, p. 375-422.
- [25] R. SJAMAAR, Symplectic reduction and Riemann-Roch formulas for multiplicities, *Bull. Amer. Math. Soc.* **33**, 1996, p. 327-338.
- [26] Y. TIAN, W. ZHANG, An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg, *Invent. Math.*, **132**, 1998, p. 229-259.
- [27] M. VERGNE, Multiplicity formula for geometric quantization, Part I, Part II, and Part III, *Duke Math. Journal*, **82**, 1996, p. 143-179, p 181-194, p 637-652.
- [28] M. VERGNE, Quantification géométrique et réduction symplectique, *Séminaire Bourbaki* **888**, 2001.
- [29] M. VERGNE, Applications of Equivariant Cohomology, International Congress of Mathematicians 2006, Vol. I, Eur. Math. Soc., Zürich, 2007, p. 635-664. Arxiv: math/0607389.
- [30] J. WEITSMAN, Non-abelian symplectic cuts and the geometric quantization of noncompact manifolds. EuroConférence Moshé Flato 2000, Part I (Dijon). *Lett. Math. Phys.*, **56**, 2001, no. 1, p. 31-40.
- [31] A. WEINSTEIN, Lecture on symplectic manifold, CBMS Regional Conf. Series in Math., **29**, 1983.

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